MAPS OF CERTAIN ALGEBRAIC CURVES INVARIANT UNDER CYCLIC INVOLUTIONS OF PERIODS THREE, FIVE, AND SEVEN

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

January, 1953

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. W. R. Hutcherson, who, as chairman of his supervisory committee, not only suggested the problem but also gave so generously of his time and energy. Without his continued interest and encouragement this dissertation would not have been possible.

The author also wishes to express his appreciation to Dr. F. W. Kokomoor, Dr. C. B. Smith, Dr. G. B. Lang, and Dr. J. S. Johnson, for serving on his supervisory committee and for their assistance in editing this dissertation.

TABLE OF CONTENTS

TABLE OF CONTENTS
(continued)

CHAPTER I

INTRODUCTION

An attempt has been made in this dissertation to study the image surfaces of certain plane and space curves containing cyclic involutions of periods three, five, and seven. The references listed under the bibliography at the end of the dissertation are numbered. Bracketed numbers throughout the text refer to the corresponding references in the bibliography.

Homogeneous projective coordinates are used entirely, and the readers are referred to the references [4, 14] for detailed explanations of same.

It is assumed, further, that the reader is familiar with the usual terminology, which is employed in this dissertation. Wherever the term "adjacent point" is used, it is understood to have the same meaning as a point in the "first order neighborhood" [10].

A (1,1) correspondence can, in general, be set up between the points of two planes by the use of a quadratic transformation and its inverse,

$$U, \qquad x_1 : x_2 : x_3 = z_1^2 : z_1 z_2 : z_2 z_3$$

and

$$U^{-1}, \qquad z_1 : z_2 : z_3 = x_1 x_2 : x_2^2 : x_1 x_3$$

However, the base points of the reference triangles require special study.

For example, let $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$ be the base points of the reference triangle in the x plane. Let $P_1(1,0,0)$, $P_2(0,1,0)$, and $P_3(0,0,1)$ be the base points of the reference triangle in the z plane. It follows then, that

$$O(a,b,c) \overset{U^{-1}}{\sim} P(ab,\ b^2,\ ac)$$

and

$$P(ab,\ b^2,\ ac) \overset{U}{\sim} O(a^2b^2,\ ab^3,\ ab^2c) = O(a,\ b,\ c).$$

If, however, b and c are both equal to zero, then

$$O_1(1,0,0) \overset{U^{-1}}{\sim} P(0,0,0)$$

and

$$P_1(1,0,0) \overset{U}{\sim} O_1(1,0,0)$$

In other words, the homologue of O_1 is undefined in the z plane. Assuming now a general point $O(a,\ b,\ c)$ in the x plane, then the coordinates of any point Q on the line $O\ O_1$ will be

$$Q\ (1 + ka,\ kb,\ kc) \quad [15] .$$

Hence

$$Q\ (1 + ka,\ kb,\ kc) \overset{U^{-1}}{\sim} Q'\left\{(1+ka)kb, k^2b^2, (1+ka)kc\right\}$$

$$= Q'\left\{(1+ka)b,\ kb^2,\ (1+ka)c\right\}$$

If one wishes to let the point Q approach O_1 along

the line O_1O_2, the transformation, on the preceding page, reduces to

$$\lim_{k \to 0} Q((1+ka), kb, 0) \overset{U^{-1}}{\sim} Q'(b, 0, 0) = P_1(1, 0, 0).$$

It is then said that $\lim_{k \to 0} Q(1+ka, kb, 0)$ is a point in the first order neighborhood of O_1 along the direction of the line O_1O_2. This point on the x plane, then, has for its homologue in the z plane the point P_1, while the homologue of P_1 is in the x plane and is the point O_1 itself. If a quadratic transformation is applied k times, one gets the k^{th} order neighborhood (k is any positive integer).

In Chapters II, IV, and VI a discussion is given concerning the images of space curves, invariant under cyclic involutions of periods three, five, and seven. The data are primarily obtained from published papers of W. R. Hutcherson [6, 7, 8].

Chapters III, V, and VII treat the image surfaces of plane curves of orders three, five, and seven, invariant under cyclic involutions of periods three, five, and seven respectively.

The material for the study in Chapter III is taken from M. L. Godeaux [2], and the material in Chapter V is taken from M^{lle} J. Dessart [1].

Most of the material in Chapter VII is original,

as far as the author has been able to determine.

It is an attempt on the part of the author to extend the findings of M^{lle} J. Dessart [1] to the case of mapping plane seventh order curves, invariant under an involution of period seven, over onto a surface of order seven in a space of five dimensions.

CHAPTER II

MAPPING OF CERTAIN SPACE CURVES INVARIANT UNDER I_3

Consider [8] a surface F_7 in S_3, invariant under the homography

$$T, \quad x_1' : x_2' : x_3' : x_4' = Ex_1 : E^2x_2 : x_3 : x_4 \quad (E^3 = 1).$$

Its equation is

$$F_7(x_1, x_2, x_3, x_4) \equiv ax_2^6x_1 + bx_1 + cx_1^5x_2 + dx_2^2 + ex_1^2x_2 + fx_1x_2^3 = 0.$$

The coefficients a, b, c, d, e, and f are assumed to be homogeneous polynomials in x_3 and x_4 of degrees 0, 6, 1, 5, 4, and 3 respectively.

The collineation T possesses the invariant points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, and the line P_3P_4 of invariant points, where the coordinates of P_3 and P_4 are respectively (0,0,1,0) and (0,0,0,1). Any point on this line can be represented by the coordinates (0,0,-k,1), which are transformed into themselves by the transformation T.

The surface F_7 has, therefore, two isolated invariant points, P_1 and P_2, and an infinite number of invariant points on the line P_3P_4.

P_2 is a simple point on F_7, while P_1 is a double point on this surface. The discussion in this chapter will be restricted to the simple point P_2.

Consider a curve C on F_7, not transformed into itself by the transformation T, but passing through P_2. The pencil of planes through the line P_1P_2 is represented by the equation

$$x_3 + px_4 = 0.$$

Of that pencil, select the plane p_1, that is tangent to the above curve C at P_2. This is the plane determined by line $x_3 = x_4 = 0$ and the tangent line to the curve C at P_2. The equation of this plane is

$$x_3 + p_1x_4 = 0.$$

It is invariant under T. The curve cut out on F_7 by this plane is likewise invariant, since both this plane and the surface are each invariant.

Now, curve C goes into curve C', since curve C is not invariant, by assumption. However, the point adjacent to P_2 along the tangent to C does go into itself along this same tangent.

Since P_2 is a simple point, and since C is any non-invariant curve through P_2, every direction through P_2, in the tangent plane to the surface F_7 at this point, is invariant under T.

This is the necessary and sufficient condition for a perfect point [10]. Thus, P_2 is a perfect point on F_7.

A general cubic surface may be written as

$$f_3 = a_{111}x_1^3 + a_{222}x_2^3 + a_{333}x_3^3 + a_{444}x_4^3 + a_{112}x_1^2x_2 +$$

$$a_{113}x_1^2x_3 + a_{114}x_1^2x_4 + a_{223}x_2^2x_3 + a_{224}x_2^2x_4 +$$

$$a_{221}x_2^2x_1 + a_{334}x_3^2x_4 + a_{331}x_3^2x_1 + a_{332}x_3^2x_2 +$$

$$a_{441}x_4^2x_1 + a_{442}x_4^2x_2 + a_{443}x_4^2x_3 + a_{123}x_1x_2x_3 +$$

$$a_{124}x_1x_2x_4 + a_{134}x_1x_3x_4 + a_{234}x_2x_3x_4 = 0.$$

This equation has twenty terms in it. Consider the system of curves cut on F_7 by these cubic surfaces. In general, this system A is not invariant. Certain points of two curves are coincident when the three equations

$$F_7 = 0,\ f_{3,1} = 0,\ f_{3,2} = 0$$

are satisfied ($f_{3,1}$ and $f_{3,2}$ are two distinct surfaces of the system f_3).

There are $7 \times 3 \times 3 = 63$ variable points depending upon the two curves involved. Since there are nineteen arbitrary constants in the equation of the general cubic surface, the curves of system $|A|$ are of dimension 19.

There are three systems of cubic surfaces that are invariant under T.

They are

$$(A_1)\quad a_{111}x_1^3 + a_{222}x_2^3 + a_{123}x_1x_2x_3 + a_{124}x_1x_2x_4 + a_{333}x_3^3 +$$
$$a_{444}x_4^3 + a_{334}x_3^2x_4 + a_{443}x_3x_4^2 = 0$$

$$(A_2)\quad a_{112}x_1^2x_2 + a_{331}x_1x_3^2 + a_{441}x_1x_4^2 + a_{134}x_1x_3x_4 +$$
$$a_{223}x_2^2x_3 + a_{224}x_2^2x_4 = 0$$

$$(A_3)\quad a_{221}x_1x_2^2 + a_{332}x_2x_3^2 + a_{442}x_2x_4^2 + a_{234}x_2x_3x_4 +$$
$$a_{113}x_1^2x_3 + a_{114}x_1^2x_4 = 0.$$

The three systems of curves on F_7 are designated by $|A_1|$, $|A_2|$, and $|A_3|$ respectively. The system (A_1) of cubic surfaces does not pass through the invariant points on F_7.

Refer projectively the curves $|A_1|$ to the hyperplanes of a linear space of seven dimensions, following the method outlined by L. Godeaux in his article published in "Bulletin de la Academie Royale de Belgique, pp. 524-543, 1927. The new surface $\emptyset$, the image of I_3, is of order 21 (three points of two curves go into one point of two image curves and therefore, 1/3 of 63 = 21).

The equations of the transformation for mapping I_3 upon $\emptyset$ in S_7 are

$$pX_1 = x_1^3 \qquad pX_5 = x_1x_2x_3$$

$$pX_2 = x_2^3 \qquad pX_6 = x_1x_2x_4$$

$$pX_3 = x_3^3 \qquad pX_7 = x_3^2x_4$$

$$pX_4 = x_4^3 \qquad pX_8 = x_3x_4^2$$

Eliminating p, x_1, x_2, x_3, and x_4 between these equations, where x_1, x_2, x_3, and x_4 are also subject to the condition $F_7(x_1,x_2,x_3,x_4) = 0$ one gets the five equations of $\emptyset$ to be

$$\left\|\begin{matrix} X_3 & X_5 & X_7 & X_8 & X_5^3 \\ X_7 & X_6 & X_8 & X_4 & X_1X_2X_7 \end{matrix}\right\| = 0$$

and $\quad \Psi(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = 0$

Now, each invariant point of F_7 has a branch point for its image on $\emptyset$. Call this image of P_2 on F_7 the point P_2' on $\emptyset$. Its coordinates are

$$X_1 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$$

The curves $|A_1|$ go through P_2 when the surfaces (A_1) have $a_{222} = 0$. Now the tangent plane to F_7 at P_2 is $x_1 = 0$. Therefore, this tangent plane cuts the new system of (A_1) surfaces in the curves

$$x_1 = 0$$

$$a_{333}x_3^3 + a_{444}x_4^3 + a_{334}x_3^2x_4 + a_{344}x_3x_4^2 = 0$$

Since these curves degenerate into three lines passing through P_2, it follows that the curves $|A_1|$, passing through P_2, possess a triple point at P_2.

Since P_2 is a triple point for $|A_1|$ then P_2' on $\emptyset$ has three adjacent points, that are coincident at P_2'. These three vary with the system $|A_1|$. Hence, P_2' is the vertex of a third order tangent cone to the surface $\emptyset$.

CHAPTER III

MAPPING OF CERTAIN PLANE CUBIC CURVES INVARIANT UNDER I_3

In this chapter will be discussed the results of mapping a plane cubic system of curves over onto a surface of order three in a space of three dimensions. Such a study has been carried out by M. L. Godeaux in a paper published in 1916 [2]. Some of the findings of this paper will be noted below.

The general cubic curve

$$a_{300}x_1^3 + a_{030}x_2^3 + a_{003}x_3^3 + a_{210}x_1^2x_2 + a_{201}x_1^2x_3 +$$
$$a_{120}x_1x_2^2 + a_{021}x_2^2x_3 + a_{102}x_1x_3^2 + a_{012}x_2x_3^2 +$$
$$a_{111}x_1x_2x_3 = 0$$

is non-invariant under the plane cyclic involution

$$T, \quad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3 \quad (E^3 = 1).$$

However, this system of curves can be divided up into three different systems. Each is invariant under the above homography T. They are

$$\text{(a)} \quad v_1x_1^3 + v_2x_2^3 + v_3x_3^3 + v_4x_1x_2x_3 = 0$$

$$\text{(b)} \quad u_1x_1^2x_2 + u_2x_2^2x_3 + u_3x_1x_3^2 = 0$$

(c) $u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2 = 0$

The base points of this involution are $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$. These points are points of coincidence invariant with respect to the involution I_3.

It is noted that the system of curves (a) do not, in general, pass through these three base points. Thus the degree (number of variable points that two curves intersect in) of this system of curves is nine.

By referring projectively each one of the curves (a) to a plane in the space of three dimensions, the following transformation is used:

$$\frac{X_1}{x_1^3} = \frac{X_2}{x_2^3} = \frac{X_3}{x_3^3} = \frac{X_4}{x_1x_2x_3} .$$

The equation for the plane then becomes

$$v_1X_1 + v_2X_2 + v_3X_3 + v_4X_4 = 0.$$

When the x_i's are eliminated in the above equations, the third order image surface $\emptyset$ becomes

$$X_1X_2X_3 = X_4^3 .$$

Points O_1, O_2, and O_3 have image points on $\emptyset$. They are, respectively, $O_1'(1,0,0,0)$, $O_2'(0,1,0,0)$, and $O_3'(0,0,1,0)$.

Consider the system (a) with the coefficient v_1 equal to zero. The branch point O_1' is a double bi-planar

point with tangent planes $X_2 = 0$ and $X_3 = 0$. Likewise, it is found that O_2' and O_3' are double bi-planar branch points with tangent planes $X_1 = X_3 = 0$ and $X_1 = X_2 = 0$ respectively.

The systems of curves (b) and (c) are likewise mapped on the surface $\emptyset$. These image curves pass through the three branch points and are tangent to one of the two tangent planes at each of these branch points. These curves are twisted cubics.

It is also shown that the images of the curves (b) lie on the cones

$$u_1X_4^2 + u_2X_2X_3 + u_3X_3X_4 = 0$$

$$u_1X_1X_4 + u_2X_4^2 + u_3X_1X_3 = 0$$

$$u_1X_1X_2 + u_2X_2X_4 + u_3X_4^2 = 0$$

These are the three cones projecting the images of the curves (b) from the branch points O_1', O_2', and O_3' respectively.

M. L. Godeaux does not make a separate study of the system of curves (c) and their images on $\emptyset$. However, they have the same characteristics as (b) and its images. The image curves of (c) have for projecting cones

$$u_1X_4^2 + u_2X_2X_4 + u_3X_2X_3 = 0$$

$$u_1x_1x_3 + u_2x_4^2 + u_3x_3x_4 = 0$$

$$u_1x_1x_4 + u_2x_1x_2 + u_3x_4^2 = 0$$

It is noted further that any two curves from the systems (b) or (c) intersect not only at the three base points of the reference triangle but also at three variable points, which form a group of the involution I_3. As an example, the two particular equations

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = 0$$

and
$$x_1^2x_2 - x_2^2x_3 - 3x_3^2x_1 = 0$$

from (b), when solved simultaneously, give in addition to the three base points also the three points of intersection $(1, \sqrt[3]{4}, -\sqrt[3]{2})$, $(1, \sqrt[3]{4}\,E, -\sqrt[3]{2}\,E^2)$ and $(1, \sqrt[3]{4}\,E^2, -\sqrt[3]{2}\,E)$. These curves, then, are of degree three in the plane, and their images on the surface $\emptyset$ are of degree one, having one variable point of intersection in addition to the three branch points.

As a final conclusion M. L. Godeaux [2] states that the necessary and sufficient condition for a cubic surface to be the image of a plane cyclic involution of period three is that it possesses three ordinary bi-planar points, with the two tangent planes at these points forming the faces of a trihedral angle.

CHAPTER IV

MAPPING OF CERTAIN SPACE CURVES INVARIANT UNDER I_5

In Chapter II it was seen that invariant space curves of order twenty-one could be mapped onto a surface of order twenty-one in space of seven dimensions. In this chapter, invariant space curves of order fifteen will be referred to a linear space of eleven dimensions. A surface of order fifteen is obtained as the image of an involution of period five.

Such a study has already been made by W. R. Hutcherson [7] in 1931. Some of his results will be reviewed below.

Consider the surface

$$F_3(x_1, x_2, x_3, x_4) \equiv ax_1^2x_3+bx_2^2x_1+cx_3^2x_4+dx_4^2x_2 = 0$$

invariant under the cyclic collineation

$$T, \quad x_1' : x_2' : x_3' : x_4' = x_1 : Ex_2 : E^2x_3 : E^3x_4, \ (E^5=1).$$

The surface F_3 has only the four invariant points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, $P_3(0,0,1,0)$, and $P_4(0,0,0,1)$.

Consider a curve C which is not invariant under T, but passing through the point P_1. Take a plane $x_3+kx_4=0$ of the pencil of planes passing through P_1 and P_2, tangent

to C. This plane is non-invariant under T and intersects the surface F_3 in a non-invariant curve C_1. It follows that the common tangent to the curves C and C_1 at P_1 is not transformed into itself. Hence, the two curves C_1 and its image C_1' do not touch each other at P_1. But C was any non-invariant curve through P_1, so it follows that P_1 is a non-perfect point of coincidence. Since a similar argument can be applied to the points P_2, P_3, and P_4, the following theorem has been proved:

THEOREM 1. The I_5 belonging to F_3 in S_3 has four non-perfect points of coincidence.

Consider next the complete system of curves $|A|$ cut out on F_3 by the quintic surfaces (A).

This system has the dimension 55, and any two members of the system intersect in 75 variable points. The complete system $|A|$ is, in general, non-invariant under T. However, it can be separated into five partial systems which are each invariant.

One of the five partial systems is deprived of points of coincidence. It will be designated by $|A_1|$, and its equations are

$$a_{5000}x_1^5 + a_{0500}x_2^5 + a_{0050}x_3^5 + a_{0005}x_4^5 + a_{1103}x_1x_2x_4^3 +$$

$$a_{1310}x_1x_2^3x_3 + a_{1022}x_1x_3^2x_4^2 + a_{2120}x_1^2x_2x_3^2 + a_{2201}x_1^2x_2^2x_4 +$$

$$a_{3011}x_1^3x_3x_4 + a_{0131}x_2x_3^3x_4 + a_{0212}x_2^2x_3x_4^2 = 0,$$

and $$F_3(x_1, x_2, x_3, x_4) = 0$$

The curves $|A_1|$ are now referred projectively to a linear space of eleven dimensions by means of the transformations

$$\begin{array}{ll} pX_1 = x_1^5 & pX_7 = x_1x_3^2x_4^2 \\ pX_2 = x_2^5 & pX_8 = x_1^2x_2x_3^2 \\ pX_3 = x_3^5 & pX_9 = x_1^2x_2^2x_4 \\ pX_4 = x_4^5 & pX_{10} = x_1^3x_3x_4 \\ pX_5 = x_1x_2x_4^3 & pX_{11} = x_2x_3^3x_4 \\ pX_6 = x_1x_2^3x_3 & pX_{12} = x_2^2x_3x_4^2 \end{array}$$

When p, x_1, x_2, x_3, and x_4 are eliminated from these twelve equations and from the equation for the surface F_3, a surface $\emptyset$, of order fifteen, is obtained. Its equations are

$$\left\| \begin{matrix} X_4 & X_5 & X_{12} \\ X_5 & X_9 & X_6 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} X_2 & X_9 & X_{12} \\ X_6 & X_{10} & X_7 \end{matrix} \right\| = 0$$

$$\left\| \begin{matrix} X_1 & X_{10} & X_8 \\ X_{10} & X_7 & X_{11} \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} X_3 & X_{11} & X_7 \\ X_{11} & X_{12} & X_5 \end{matrix} \right\| = 0,$$

and $$aX_8 + bX_6 + cX_{11} + dX_{12} = 0.$$

To P_1 on F_3 corresponds on $\emptyset$ the branch point P_1', with coordinates $X_1 = 1$, and $X_i = 0$, where $i = 2, 3, \ldots 12$.

Suppose now that the coefficient a_{5000} of (A_1) equals zero. Then the new system (A_{11}) of quintic surfaces passes through P_1, which is a simple point for the surface F_3 with tangent plane $x_3 = 0$. The system (A_{11}) cuts $x_3 = 0$ in the curves

$$a_{0500}x_2^5 + a_{0005}x_4^5 + a_{1103}x_1x_2x_4^3 + a_{2201}x_1^2x_2^2x_4 = 0$$

and $$x_3 = 0.$$

For general values of the constants, this is a quintic curve with a triple point at P_1, two branches being tangent to the line $x_2 = x_3 = 0$ and one to the line $x_3 = x_4 = 0$.

In the plane $x_3 = 0$ the involution I_5 is generated by the homography T_1, which is

$$x_1' : x_2' : x_4' = x_1 : Ex_2 : E^3x_4 .$$

By use of the plane [3] quadratic transformations

$$S, \qquad x_1 : x_2 : x_4 = z_1^2 : z_1z_2 : z_2z_4$$

and S^{-1}, $\quad z_1 : z_2 : z_4 = x_1x_2 : x_2^2 : x_1x_4$,

the characteristics of the adjacent invariant points along the two invariant directions at P_1 are investigated. By the application of

$$ST_1S^{-1} \equiv T_1'$$

$$(z_1,z_2,z_4) \overset{S^{-1}}{\sim} (x_1x_2,x_2^2,x_1x_4) \overset{T_1}{\sim}$$

$$(Ex_1x_2,E^2x_2^2,E^3x_1x_4) \overset{S}{\sim} (z_1,Ez_2,E^2z_4) .$$

Thus the new transformation is

$$T_1', \qquad x_1' : x_2' : x_4' = x_1 : Ex_2 : E^2x_4 .$$

The invariant point adjacent to P_1 along the line $x_3 = x_4 = 0$ is still a non-perfect point of coincidence. By applying the transformation

$$T_1'' = ST_1'S^{-1},$$

one gets

$$(z_1,z_2,z_4) \sim (z_1,Ez_2,Ez_4) .$$

Thus, the point in the second order neighborhood of P_1 along $x_3 = x_4=0$ is a perfect point of coincidence .

By use of the transformation

$$R, \quad x_1 : x_2 : x_4 = z_1^2 : z_2 z_4 : z_1 z_4$$

and R^{-1}, $\quad z_1 : z_2 : z_4 = x_1 x_4 : x_1 x_2 : x_4^2$,

it is shown that there is a perfect point of coincidence in the first order neighborhood of A_1 along $x_2 = x_3 = 0$. The following theorem has, therefore, been proved:

THEOREM 2. The non-perfect coincidence point P_1 on F_3 has one adjacent perfect point along the line $x_2 = x_3 = 0$, a non-perfect one along the line $x_3 = x_4 = 0$, with a perfect one adjacent to this.

In a similar manner the neighborhoods of P_2, P_3, and P_4 are investigated with the following results noted:

THEOREM 3. The non-perfect coincidence point P_2 on F_3 has one adjacent perfect point along the line $x_1 = x_4 = 0$, a non-perfect adjacent one along $x_1 = x_3 = 0$, with a perfect one adjacent to this.

THEOREM 4. The imperfect coincidence point P_3 on F_3 has one adjacent perfect point of coincidence along the line $x_1 = x_4 = 0$. There is also a perfect point in the domain of the second order of P_3 along the line $x_2 = x_4 = 0$.

Note. Theorem 4, just stated above, replaces Theorem 7 in the reference [7] . More accurate results

were obtained by revising the transformations V and V^{-1} to read

$$V_1, \qquad y_1 : y_2 : y_3 = w_1w_2 : w_1w_3 : w_3^2$$

and

$$V_1^{-1}, \qquad w_1 : w_2 : w_3 = y_2^2 : y_1y_3 : y_2y_3$$

THEOREM 5. _The non-perfect coincidence point_ $\underline{P_4}$ _has an adjacent perfect point along the line_ $\underline{x_2} = \underline{x_3} = \underline{0}$, _a non-perfect one along the line_ $\underline{x_1} = \underline{x_2} = \underline{0}$, _with a perfect one adjacent to this._

CHAPTER V

MAPPING OF CERTAIN PLANE QUINTIC CURVES INVARIANT UNDER I_5

In Chapter III some of the results of mapping invariant cubic plane curves over onto a surface in space of three dimensions were given. It is now proposed to investigate the mapping of invariant plane quintic curves onto a surface in a space of four dimensions. Such a study has already been made by M[lle] J. Dessart [1]. In this paper the singularities of the branch points of a surface F in S_4 are studied. The surface is obtained by mapping a plane cyclic collineation I_5 onto a surface in a space of four dimensions. The homography employed is

$$T, \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^{\alpha} x_3,$$

where $E^5 = 1$, and $\alpha = 2$, 3, or 4.

By suitable choice of primitive fifth roots of unity, it is shown that the totality of results is the same, whether α has the value 2, 3, or 4. The study can therefore be limited to the case where $\alpha = 2$.

The following fifth order plane curves, invariant under the homography T, are used:

$|C_0|$, $v_0x_1^5 + v_1x_1^2x_2x_3^2 + v_2x_1x_2^3x_3 + v_3x_2^5 + v_4x_3^5 = 0$

$|C_1|$, $u_0x_1^3x_3^2 + u_1x_1^2x_2^2x_3 + u_2x_1x_2^4 + u_3x_2x_3^4 = 0$

$|C_2|$, $u_0x_1^3x_2x_3 + u_1x_1^2x_2^3 + u_2x_1x_3^4 + u_3x_2^2x_3^3 = 0$

$|C_3|$, $u_0x_1^4x_3 + u_1x_1^3x_2^2 + u_2x_1x_2x_3^3 + u_3x_2^3x_3^2 = 0$

$|C_4|$, $u_0x_1^4x_2 + u_1x_1^2x_3^3 + u_2x_1x_2^2x_3^2 + u_3x_2^4x_3 = 0.$

The system of curves $|C_0|$ have no base points, while the other four systems have the base points $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$.

The $|C_0|$ curves are now, projectively, mapped over into the hyperplanes of a linear space S_4 of dimension four. The equations are

$$\frac{X_0}{x_1^5} = \frac{X_1}{x_1^2x_2x_3^2} = \frac{X_2}{x_1x_2^3x_3} = \frac{X_3}{x_2^5} = \frac{X_4}{x_3^5}$$

By eliminating x_1, x_2, and x_3 from these equations, the following surface F is obtained:

$$\left\|\begin{matrix} X_1 & X_2 & X_0X_4 \\ X_2 & X_3 & X_1^2 \end{matrix}\right\| = 0.$$

Likewise, when the curves $|C_1|$, $|C_2|$, $|C_3|$, and $|C_4|$ are mapped onto the surface F, their new equations become

$$|\Gamma_1|, \quad \left\|\begin{matrix} x_0x_4 & x_1 & x_2 & (-u_3x_4) \\ x_1^2 & x_2 & x_3 & (u_0x_1 + u_1x_2 + u_2x_3) \end{matrix}\right\| = 0$$

$$|\Gamma_2|, \quad \left\|\begin{matrix} x_0x_4 & x_1 & x_2 & (-u_3x_4) \\ x_1^2 & x_2 & x_3 & (u_0x_1 + u_1x_2 + u_2x_4) \end{matrix}\right\| = 0$$

$$|\Gamma_3|, \quad \left\|\begin{matrix} x_0x_4 & x_1 & x_2 & (-u_1x_0) \\ x_1^2 & x_2 & x_3 & (u_0x_0 + u_2x_1 + u_3x_2) \end{matrix}\right\| = 0$$

$$|\Gamma_4|, \quad \left\|\begin{matrix} x_0x_4 & x_1 & x_2 & (-u_0x_0) \\ x_1^2 & x_2 & x_3 & (u_1x_1 + u_2x_2 + u_3x_3) \end{matrix}\right\| = 0$$

To study the behavior of the curves $|C_i|$, passing through the point O_1, infinitely near this point, M^lle Dessart makes use of the transformations

$$U, \qquad x_1 : x_2 : x_3 = z_1^2 : z_1z_2 : z_2z_3$$

$$V, \qquad x_1 : x_2 : x_3 = z_1^2 : z_2z_3 : z_1z_3 \,.$$

When the transformation U is applied first, one gets the characteristics of the neighborhood points of O_1 along the invariant line $x_3 = 0$. When V is applied first, the characteristics of the neighborhood points along the invariant line $x_2 = 0$ are determined. (See page 19, Chapter IV).

Since an harmonic homology exists, the behavior of the curves $|C_i|$ $(i = 0, 1, 2, 3, 4)$ can immediately be determined at the point O_3. For the point O_2, the following two transformations are used:

$$W, \qquad x_1 : x_2 : x_3 = z_1z_3 : z_2^2 : z_2z_3$$

$$R, \qquad x_1 : x_2 : x_3 = z_1z_2 : z_2^2 : z_1z_3 \text{ .}$$

The author notes these conclusions from M[lle] Dessart's paper. The image of the plane cyclic involution I_5 is a surface of order five in a space of four dimensions. This surface has two triple points, whose tangent cones consist of a second order cone and a plane. The surface has also one double bi-planar point, which is infinitely near one double bi-planar point.

There exists also on this surface four systems of twisted fifth order curves each consisting of ∞^3 different curves. The curves belonging to the first $|\Gamma_1|$ of these four systems pass doubly through one of the two triple points, with the two tangent lines at this point lying in the quadric cone tangent to the surface at this same point; the curves pass simply through the other triple point with its tangent in the plane, which is tangent to the surface at that point. Finally, the same curves pass simply through the double bi-planar point with the tangent

line at this point lying in one of the two tangent planes at the point.

A second system $|\Gamma_4|$ of these twisted fifth order curves behaves exactly as did this first system just described. However, the roles of the two triple points, and of the two tangent planes at the double bi-planar point, have been reversed.

A third system $|\Gamma_2|$ of these curves passes doubly through one of the triple points, with one tangent line lying in the quadric cone and the other in the tangent plane. A curve of this system passes simply through the other triple point with its tangent line on the cone at that point. It also passes simply through the double bi-planar point with its tangent line being the intersection of the two tangent planes to the surface at this point.

If the fifth order surface in question is projected from the double bi-planar point onto a hyperplane, a cubic surface is obtained, having an ordinary double bi-planar point infinitely near the center of the projection. The projections of the third system of fifth order twisted curves on this new surface are now of order four and pass simply through this double bi-planar point, touching one of the tangent planes to the surface at this point.

Finally the fourth and last system $|\Gamma_3|$ of the

fifth order twisted curves has the same properties as the curves belonging to the third system. The roles of the triple points and of the tangent planes at the double bi-planar point of the cubic surface are merely reversed.

CHAPTER VI

MAPPING OF CERTAIN SPACE CURVES INVARIANT UNDER I_7

The author has already investigated some mappings of space curves invariant under I_3 and I_5 in chapters II and IV. It now seems desirable also to investigate some mappings of space curves invariant under an involution of period seven. W. R. Hutcherson [6] in his 1934 paper discusses the involution I_7 belonging to F_3 in S_3. Consider the surface

$$F_3(x_1,x_2,x_3,x_4) \equiv ax_2^2x_3 + bx_3^2x_1 + cx_1x_2x_4 = 0$$

in S_3, invariant under the cyclic collineation

$$T, \quad x_1' : x_2' : x_3' : x_4' = x_1 : Ex_2 : E^2x_3 : E^3x_4, \ (E^7=1).$$

The four invariant points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, $P_3(0,0,1,0)$, and $P_4(0,0,0,1)$ are shown to be non-perfect points of coincidence by the same method as already explained in Chapter IV.

The cubic surface F_3 is cut successively by invariant surfaces of orders seven, six, five, four, and three. The resultant systems of curves are then mapped

over onto surfaces in other spaces. It will be the purpose of this chapter to merely list the major results obtained. The reader is referred to the original paper [6], for more detailed information.

Sections by Septics.

The partial system (A_1) of invariant septic surfaces, is

$$a_1x_1^7 + a_2x_2^7 + a_3x_3^7 + a_4x_4^7 + a_5x_1^4x_2x_4^2 + a_6x_1^4x_3^2x_4 +$$

$$a_7x_1^3x_2^2x_3x_4 + a_8x_1^3x_2x_3^3 + a_9x_1^2x_3x_4^4 + a_{10}x_1^2x_2^4x_4 +$$

$$a_{11}x_1^2x_2^3x_3^2 + a_{12}x_1x_2x_3^2x_4^3 + a_{13}x_1x_2^2x_4^4 + a_{14}x_1x_3^4x_4^2 +$$

$$a_{15}x_1x_2^5x_3 + a_{16}x_2x_3^5x_4 + a_{17}x_2^2x_3^3x_4^2 + a_{18}x_2^3x_3x_4^3 = 0.$$

This surface is referred projectively to a linear space of seventeen dimensions, using

$$\begin{aligned}
pX_1 &= x_1^7\ , & pX_{10} &= x_1^2x_2^4x_4\ ,\\
pX_2 &= x_2^7\ , & pX_{11} &= x_1^2x_2^3x_3^2\ ,\\
pX_3 &= x_3^7\ , & pX_{12} &= x_1x_2x_3^2x_4^3\ ,\\
pX_4 &= x_4^7\ , & pX_{13} &= x_1x_2^2x_4^4\ ,\\
pX_5 &= x_1^4x_2x_4^2\ , & pX_{14} &= x_1x_3^4x_4^2\ ,\\
pX_6 &= x_1^4x_3^2x_4\ , & pX_{15} &= x_1x_2^5x_3\ ,\\
pX_7 &= x_1^3x_2^2x_3x_4\ , & pX_{16} &= x_2x_3^5x_4\ ,\\
pX_8 &= x_1^3x_2x_3^3\ , & pX_{17} &= x_2^2x_3^3x_4^2\ ,\\
pX_9 &= x_1^2x_3x_4^4\ , & pX_{18} &= x_2^3x_3x_4^3\ .
\end{aligned}$$

When p, x_1, x_2, x_3, and x_4 are eliminated from these equations and from $F_3 = 0$, one obtains a surface of order twenty-one. It is

$$\left\|\begin{matrix} X_1 & X_5 & X_8 & X_7 & X_6 \\ X_5 & X_{13} & X_{17} & X_{18} & X_{12} \end{matrix}\right\| = 0 ,$$

$$\left\|\begin{matrix} X_2 & X_{10} & X_{11} & X_{18} & X_{15} \\ X_{10} & X_5 & X_6 & X_9 & X_7 \end{matrix}\right\| = 0 ,$$

$$\left\|\begin{matrix} X_3 & X_{14} & X_{17} & X_{16} \\ X_{14} & X_9 & X_{13} & X_{12} \end{matrix}\right\| = 0 ,$$

$$\left\|\begin{matrix} X_4 & X_{13} & X_{12} \\ X_9 & X_7 & X_8 \end{matrix}\right\| = 0 , \qquad \left\|\begin{matrix} X_{18} & X_{12} \\ X_{17} & X_{14} \end{matrix}\right\| = 0 ,$$

and $aX_{17} + bX_{14} + cX_{12} = 0$.

The following theorems are proved:

THEOREM 1. _Along the invariant direction $x_1=x_3=0$, the invariant point P_2 has an imperfect point in the first-order neighborhood and a perfect one in the second-order neighborhood._

THEOREM 2. _Along the invariant direction $x_3=x_4=0$, the invariant imperfect point P_2 has an imperfect point in the first-order neighborhood and a perfect one in the_

<u>second-order neighborhood</u>.

THEOREM 3. *The imperfect point P_2 on F_3 has no perfect points in the neighborhood of the first order, but precisely two perfect ones in the neighborhood of the second order.*

THEOREM 4. *Along the invariant direction $x_1 = x_2 = 0$, the invariant imperfect point P_3 has an imperfect adjacent point and a perfect one in the neighborhood of the second order.*

THEOREM 5. *Along the invariant direction $x_1 = x_4 = 0$, the invariant imperfect point P_3 has no perfect points in the first and second-order neighborhoods but does have one in the third-order neighborhood.*

The last four sections will be illustrated by merely listing the equations used and resulting theorems.

<u>Sections by Sextics</u>.

The sextic surfaces used are

$$(B_1), \quad b_1x_1x_4^5 + b_2x_2x_3x_4^4 + b_3x_3^3x_4^3 + b_4x_1^2x_2^2x_4^2 + b_5x_1^3x_3x_4^2 + b_6x_2^5x_4 + b_7x_1x_2^3x_3x_4 + b_8x_1^2x_2x_3^2x_4 + b_9x_1^2x_3^4 + b_{10}x_1x_2^2x_3^3 + b_{11}x_2^4x_3^2 + b_{12}x_1^5x_2 = 0 .$$

The image surface ϕ is

$$\left\|\begin{matrix} x_1 & x_4 & x_7 \\ x_2 & x_7 & x_{11} \end{matrix}\right\| = 0, \qquad \left\|\begin{matrix} x_4 & x_7 & x_6 \\ x_8 & x_{10} & x_{11} \end{matrix}\right\| = 0,$$

$$\left\|\begin{matrix} x_7 & x_{10} & x_6 \\ x_8 & x_9 & x_7 \end{matrix}\right\| = 0, \qquad \left\|\begin{matrix} x_5 & x_2 & x_3 \\ x_{12} & x_4 & x_8 \end{matrix}\right\| = 0,$$

and $$aX_7 + bX_8 + cX_4 = 0 .$$

THEOREM 6. *The $|B_1|$ curves pass through the imperfect points only along the invariant directions.*

Sections by Quintics.

The quintic surfaces used are

$$(C_1), \quad c_1x_1^2x_4^3 + c_2x_1x_2x_3x_4^2 + c_3x_2^3x_4^2 + c_4x_1x_3^3x_4 + c_5x_2^2x_3^2x_4 + c_6x_2x_3^4 + c_7x_1^4x_3 + c_8x_1^3x_2^2 = 0 .$$

The image surface ϕ is

$$\left\|\begin{matrix} x_1 & x_2 & x_4 \\ x_2 & x_5 & x_6 \end{matrix}\right\| = 0,$$

$$\left\|\begin{matrix} x_3 & x_5 \\ x_5 & x_6 \end{matrix}\right\| = 0, \qquad \left\|\begin{matrix} x_2 & x_7 \\ x_3 & x_8 \end{matrix}\right\| = 0,$$

and $$aX_5 + bX_4 + cX_2 = 0 .$$

THEOREM 7. *The $|C_1|$ curves pass through imperfect*

points only along invariant directions.

Sections by Quartics.

The quartic surfaces are

$$(D_1), \quad d_1x_2x_4^3 + d_2x_3^2x_4^2 + d_3x_1^3x_4 + d_4x_1x_2^3 + d_5x_1^2x_2x_3 = 0,$$

and the resulting image surface is

$$X_1X_5^2 = X_2X_3X_4$$

and $$aX_1X_5 + bX_2X_3 + cX_1X_3 = 0.$$

THEOREM 8. The $|D_1|$ curves pass through imperfect points only along invariant directions.

Sections by Cubics.

Finally, the cubic surfaces are

$$(E_1), \quad e_1x_2^2x_3 + e_2x_1x_3^2 + e_3x_1x_2x_4 = 0,$$

while the ϕ surface becomes the line

$$aX_1 + bX_2 + cX_3 = 0 .$$

This is an illustration of an I_7 contained on a surface whose image is not a two dimensional geometric entity (surface), but instead is a one dimensional entity (line).

THEOREM 9. The system of invariant curves cut out upon F_3 by surfaces of degree lower than seven all

<u>pass</u> <u>through</u> <u>the</u> <u>coincidence</u> <u>points</u> <u>along</u> <u>the</u> <u>invariant</u> <u>directions</u>. <u>The</u> <u>number</u> <u>of</u> <u>branches</u> <u>through</u> <u>each</u> <u>point</u> <u>is</u> <u>less</u> <u>than</u> <u>seven</u>.

CHAPTER VII

MAPPING OF CERTAIN PLANE SEPTIC CURVES INVARIANT UNDER I_7

Invariant Curves.

It has been seen in chapters three and five the results obtained by mapping third and fifth order plane algebraic curves, invariant under cyclic involutions of periods three and five respectively, over onto spaces of higher dimensions. In this final chapter the writer will show, using the methods developed by M. L. Godeaux [2, 3] and M^{lle} J. Dessart [1], the results of mapping a system of seventh order plane algebraic curves, invariant under the cyclic involution

$$T, \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3 , \quad E^7 = 1$$

onto a linear space S_5, of five dimensions.

The homography T has three invariant points. They are $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$.

The general system of plane curves of order seven is

$$(1)\quad a_{700}x_1^7 + a_{610}x_1^6x_2 + a_{601}x_1^6x_3 + a_{511}x_1^5x_2x_3 +$$
$$a_{502}x_1^5x_3^2 + a_{520}x_1^5x_2^2 + a_{403}x_1^4x_3^3 + a_{412}x_1^4x_2x_3^2 +$$
$$a_{421}x_1^4x_2^2x_3 + a_{430}x_1^4x_2^3 + a_{340}x_1^3x_2^4 + a_{304}x_1^3x_3^4 +$$
$$a_{331}x_1^3x_2^3x_3 + a_{322}x_1^3x_2^2x_3^2 + a_{313}x_1^3x_2x_3^3 + a_{232}x_1^2x_2^3x_3^2 +$$
$$a_{241}x_1^2x_2^4x_3 + a_{250}x_1^2x_2^5 + a_{205}x_1^2x_3^5 + a_{223}x_1^2x_2^2x_3^3 +$$
$$a_{214}x_1^2x_2x_3^4 + a_{151}x_1x_2^5x_3 + a_{160}x_1x_2^6 + a_{106}x_1x_3^6 +$$
$$a_{115}x_1x_2x_3^5 + a_{124}x_1x_2^2x_3^4 + a_{142}x_1x_2^4x_3^2 + a_{133}x_1x_2^3x_3^3 +$$
$$a_{016}x_2x_3^6 + a_{025}x_2^2x_3^5 + a_{034}x_2^3x_3^4 + a_{043}x_2^4x_3^3 +$$
$$a_{061}x_2^6x_3 + a_{052}x_2^5x_3^2 + a_{070}x_2^7 + a_{007}x_3^7 = 0.$$

It is, in general, non-invariant under the transformation T.

This equation can be split up, however, into seven equations, which are each invariant. They are

$$|C_0|,\quad v_0x_1^7+v_1x_1^3x_2x_3^3+v_2x_1^2x_2^3x_3^2+v_3x_1x_2^5x_3+v_4x_2^7+v_5x_3^7 = 0$$

$$|C_1|,\quad u_0x_1^4x_3^3+u_1x_1^3x_2^2x_3^2+u_2x_1^2x_2^4x_3+u_3x_1x_2^6+u_4x_2x_3^6 = 0$$

$$|C_2|,\quad u_0x_1^4x_2x_3^2+u_1x_1^3x_2^3x_3+u_2x_1^2x_2^5+u_3x_1x_3^6+u_4x_2^2x_3^5 = 0$$

$$|C_3|,\quad u_0x_1^5x_3^2+u_1x_1^4x_2^2x_3+u_2x_1^3x_2^4+u_3x_1x_2x_3^5+u_4x_2^3x_3^4 = 0$$

$$|C_4|,\quad u_0x_1^5x_2x_3+u_1x_1^4x_2^3+u_2x_1^2x_3^5+u_3x_1x_2^2x_3^4+u_4x_2^4x_3^3 = 0$$

$$|C_5|,\quad u_0x_1^6x_3+u_1x_1^5x_2^2+u_2x_1^2x_2x_3^4+u_3x_1x_2^3x_3^3+u_4x_2^5x_3^2 = 0$$

$$|C_6|,\quad u_0x_1^6x_2+u_1x_1^3x_3^4+u_2x_1^2x_2^2x_3^3+u_3x_1x_2^4x_3^2+u_4x_2^6x_3 = 0$$

Refer now, projectively, the curve $|C_0|$ to a linear space S_5 of five dimensions using the transformation

$$(2)\qquad \frac{X_0}{x_1^7} = \frac{X_1}{x_1^3x_2x_3^3} = \frac{X_2}{x_1^2x_2^3x_3^2} = \frac{X_3}{x_1x_2^5x_3} = \frac{X_4}{x_2^7} = \frac{X_5}{x_3^7}$$

By eliminating the x_i's from these equations, one gets for the new surface F the equations

$$(3)\qquad \left\|\begin{matrix} X_1 & X_2 & X_3 & X_0X_5 \\ X_2 & X_3 & X_4 & X_1^2 \end{matrix}\right\| = 0\,,$$

or

(a) $\quad X_1X_3 = X_2^2$

(b) $\quad X_2X_4 = X_3^2$

(c) $\quad X_1^2X_3 = X_0X_4X_5$

These three equations are independent. From them three dependent equations are obtained also. These help

one in determining the surface F. They are

(d) $x_1x_4 = x_2x_3$

(e) $x_1^3 = x_0x_2x_5$

(f) $x_1^2x_2 = x_0x_3x_5$

Order of a Surface.

To determine the order of the surface F, the method of F. S. Woods [15] will be used. Briefly, the method consists of determining the number of points of intersection the surface makes with two general cutting hyperplanes. In other words, if the surface is represented by n-2 equations in a space of n dimensions, it becomes necessary to solve n equations simultaneously. In this case, the equations to be solved are

1. $x_1x_3 = x_2^2$
2. $x_2x_4 = x_3^2$
3. $x_1^2x_3 = x_0x_4x_5$
4. $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$
5. $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 = 0.$

By elimination of variables, the five equations are reduced to one equation of degree seven. Thus there are seven

points of intersection, and the surface F is of order seven. It may be appropriate to remark, here, that in the process of solution, several variables may be cancelled out without loss of roots; as the coordinates of the resulting points will, in general, not satisfy all of the equations involved including the dependent equations (d), (e), and (f).

<u>Harmonic</u> <u>Homology</u>.

A careful study of the invariant curves $|C_1|$, shows that the homography

$$\Omega, \qquad y_1 : y_2 : y_3 = x_3 : x_2 : x_1$$

in the plane containing the involution I_7, is a harmonic homology of center $A(x_2 = 0,\ x_1 + x_3 = 0)$ and with axis a $(x_1 - x_3 = 0)$. This homology transforms O_1 into O_3, O_3 into O_1, and O_2 into itself. Furthermore, the homology also transforms the totality of curves $|C_0|$ into $|C_0|$, $|C_1|$ into $|C_6|$, $|C_2|$ into $|C_5|$, and $|C_3|$ into $|C_4|$.

This harmonic homology corresponds in S_5 to the harmonic homology

$$\Omega', \qquad \frac{X_0}{Y_5} = \frac{X_1}{Y_1} = \frac{X_2}{Y_2} = \frac{X_3}{Y_3} = \frac{X_4}{Y_4} = \frac{X_5}{Y_0}$$

with center at A' $(X_1 = X_2 = X_3 = X_4 = 0,\ X_0 + X_5 = 0)$

and axis the hyperplane a' $(X_0 - X_5 = 0)$. This homology transforms the surface F into itself.

Image Curves.

We will designate by $|\Gamma_0|$ the hyperplane sections of F, which correspond to the curves $|C_0|$. Likewise, $|\Gamma_1|$, $|\Gamma_2|$, $|\Gamma_3|$, $|\Gamma_4|$, $|\Gamma_5|$, and $|\Gamma_6|$ are the curves on F, which correspond, respectively, to the curves $|C_1|$, $|C_2|$, $|C_3|$, $|C_4|$, $|C_5|$, and $|C_6|$.

By the indicated homology, then, these curves are transformed as follows: $|\Gamma_0|$ goes into itself, $|\Gamma_1|$ into $|\Gamma_6|$, $|\Gamma_2|$ into $|\Gamma_5|$, and $|\Gamma_3|$ into $|\Gamma_4|$.

It is observed that any curve from the system $|C_i|$ $(i = 1, 2, \ldots 6)$ intersects a curve from $|C_0|$ in forty-nine points, forming seven groups of the involution I_7. It follows, therefore, that $|\Gamma_i|$ $(i = 1, 2, \ldots 6)$ will intersect $|\Gamma_0|$ in seven points, making the curves $|\Gamma_i|$ of order seven.

The equations of these curves are

$$|\Gamma_1|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_4x_5) \\ x_2 & x_3 & x_4 & x_1^2 & (u_0x_1+u_1x_2+u_2x_3+u_3x_4) \end{matrix} \right\| = 0$$

$$|\Gamma_2|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_4x_5) \\ x_2 & x_3 & x_4 & x_1^2 & (u_0x_1+u_1x_2+u_2x_3+u_3x_5) \end{matrix} \right\| = 0$$

$$|\Gamma_3|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_1x_0x_1-u_2x_0x_2) \\ x_2 & x_3 & x_4 & x_1^2 & (u_3x_1^2+u_4x_1x_2+u_0x_0x_1) \end{matrix} \right\| = 0$$

$$|\Gamma_4|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_3x_1x_5-u_4x_2x_5) \\ x_2 & x_3 & x_4 & x_1^2 & (u_0x_1^2+u_1x_1x_2+u_2x_1x_5) \end{matrix} \right\| = 0$$

$$|\Gamma_5|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_1x_0) \\ x_2 & x_3 & x_4 & x_1^2 & (u_0x_0+u_2x_1+u_3x_2+u_4x_3) \end{matrix} \right\| = 0$$

$$|\Gamma_6|, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 & (-u_0x_0) \\ x_2 & x_3 & x_4 & x_1^2 & (u_1x_1+u_2x_2+u_3x_3+u_4x_4) \end{matrix} \right\| = 0$$

Branch Point O_0'.

To point O_1 on the plane corresponds on F the point $O_0'(1,0,0,0,0,0)$. The singularities of this point will now be studied. Consider the family (∞^4) of curves from the system $|C_0|$, passing through the invariant point O_1. The equation for this family of curves is

$$(4) \quad v_1x_1^3x_2x_3^3 + v_2x_1^2x_2^3x_3^2 + v_3x_1x_2^5x_3 + v_4x_2^7 + v_5x_3^7 = 0$$

Applying to equations (4) the quadratic transformation

$$U, \qquad x_1' : x_2' : x_3' = z_1^2 : z_1z_2 : z_2z_3$$

and simplifying, we get

$$(5) \quad z_1^7(v_1z_3^3 + v_2z_2z_3^2 + v_3z_2^2z_3 + v_4z_2^3) + v_5z_2^3z_3^7 = 0$$

This shows that to the point O_{12} (the first order neighborhood of O_1 on $x_3 = 0$), corresponds the triple point ($z_2 = z_3 = 0$) for the curves (5).

In order to obtain the points, infinitely near $O_0'(1,0,0,0,0,0)$ on F, which correspond to the points infinitely near the point ($z_2 = z_3 = 0$), it is necessary, first, to project the surface F from the point O_1' onto the hyperplane $X_0 = 0$. This gives the independent equations

F_1,
$$X_1X_3 = X_2^2$$
$$X_2X_4 = X_3^2$$
$$X_0 = 0$$

and the dependent equation

$$X_1X_4 = X_2X_3$$

It is noted that the plane $X_0 = X_2 = X_3 = 0$ satisfies the independent equations, but not the dependent equation. Thus F_1 must be a cubic surface [1]. (This can also be verified by the method explained earlier).

Second, apply the transformation U to

$$\frac{X_1}{x_1^3x_2x_3^3} = \frac{X_2}{x_1^2x_2^3x_3^2} = \frac{X_3}{x_1x_2^5x_3} = \frac{X_4}{x_2^7} = \frac{X_5}{x_3^7}$$

and obtain the simplified expression

$$\frac{X_1}{z_1^7 z_3^3} = \frac{X_2}{z_1^7 z_2 z_3^2} = \frac{X_3}{z_1^7 z_2^2 z_3} = \frac{X_4}{z_1^7 z_2^3} = \frac{X_5}{z_2^3 z_3^7}$$

Since one is interested in approaching the point $(z_2 = z_3 = 0)$ from all directions, let $z_3 = kz_2$ and substitute in the last equations. This gives in simplified form

$$\frac{X_1}{k^3 z_1^7} = \frac{X_2}{k^2 z_1^7} = \frac{X_3}{k z_1^7} = \frac{X_4}{z_1^7} = \frac{X_5}{k^7 z_2^7}$$

Passing to the limit as $z_2 \rightarrow 0$ must, to have meaning, make $X_5 = 0$. Thus the equations, after multiplying through by z_1^7, become

(6) $$\frac{X_1}{k^3} = \frac{X_2}{k^2} = \frac{X_3}{k} = \frac{X_4}{1}, \quad X_5 = 0$$

It follows that

$$\frac{X_1}{X_2} = k, \quad \frac{X_2}{X_3} = k, \quad \frac{X_3}{X_4} = k, \quad X_5 = 0$$

Eliminating k, one gets (as illustrated for F_1) the cubic cone

$$X_1 X_3 = X_2^2$$

$$X_2 X_4 = X_3^2$$

$$X_5 = 0,$$

which intersects the cubic surface F_1 in a twisted cubic curve

$$(\Upsilon_1) \quad \begin{aligned} X_1X_3 &= X_2^2 \\ X_2X_4 &= X_3^2 \\ X_0 &= X_5 = 0. \end{aligned}$$

This shows that the points of the first order neighborhood of the point ($z_2 = z_3 = 0$), as well as the points in the domain of the first order neighborhood of O_{12}, correspond projectively to points of this cubic curve. The points of this curve are projections of the points infinitely near O_0' and on the cubic tangent cone to F at the point O_0'.

Applying now the quadratic transformation

$$V, \qquad x_1' : x_2' : x_3' = z_1^2 : z_2z_3 : z_1z_3$$

three times, successively, to the curves (4),

$$v_1x_1^3x_2x_3^3 + v_2x_1^2x_2^3x_3^2 + v_3x_1x_2^5x_3 + v_4x_2^7 + v_5x_3^7 = 0,$$

one gets

$$v_1z_1^9z_2 + v_2z_1^6z_2^3z_3 + v_3z_1^3z_2^5z_3^2 + v_4z_2^7z_3^3 + v_5z_1^7z_3^3 = 0$$

and

$$v_1z_1^{15}z_2 + v_2z_1^{10}z_2^3z_3^3 + v_3z_1^5z_2^5z_3^6 + v_4z_2^7z_3^9 + v_5z_1^{14}z_3^2 = 0$$

and finally

$$v_1z_1^{21}z_2 + v_2z_1^{14}z_2^3z_3^5 + v_3z_1^7z_2^5z_3^{10} + v_4z_2^7z_3^{15} + v_5z_1^{21}z_3 = 0$$

or

$$(7) \quad z_1^{21}(v_1z_2 + v_5z_3) + v_2z_1^{14}z_2^3z_3^5 + v_3z_1^7z_2^5z_3^{10} + v_4z_2^7z_3^{15} = 0$$

This shows that to O_{1333} in the third order neighborhood of O_1 on $x_2 = 0$ corresponds simply the point ($z_2 = z_3 = 0$). Thus, the curves (4) pass simply through O_{13}, O_{133}, and O_{1333} on line $x_2 = 0$.

Applying the quadratic transformation V three times is equivalent to using the transformation V_3, $x_1' : x_2' : x_3' = z_1^4 : z_2 z_3^3 : z_1^3 z_3$ once.

Using this transformation on the curves of the x plane and then projecting the resultant curves of the z plane over onto S_5 gives

$$\frac{X_1}{x_1^3 x_2 x_3^3} = \frac{X_2}{x_1^2 x_2^3 x_3^2} = \frac{X_3}{x_1 x_2^5 x_3} = \frac{X_4}{x_2^7} = \frac{X_5}{x_3^7}$$

and

$$\frac{X_1}{z_1^{21} z_2} = \frac{X_2}{z_1^{14} z_2^3 z_3^5} = \frac{X_3}{z_1^7 z_2^5 z_3^{10}} = \frac{X_4}{z_2^7 z_3^{15}} = \frac{X_5}{z_1^{21} z_3}$$

Again let $z_3 = kz_2$ and substitute. One gets

$$\frac{X_1}{z_1^{21} z_2} = \frac{X_2}{k^5 z_1^{14} z_2^8} = \frac{X_3}{k^{10} z_1^7 z_2^{15}} = \frac{X_4}{k^{15} z_2^{22}} = \frac{X_5}{k z_1^{21} z_2}$$

or

$$\frac{X_1}{z_1^{21}} = \frac{X_2}{k^5 z_1^{14} z_2^7} = \frac{X_3}{k^{10} z_1^7 z_2^{14}} = \frac{X_4}{k^{15} z_2^{21}} = \frac{X_5}{k z_1^{21}}$$

When allowing z_2 to approach zero, this reduces to

(8) $\quad X_5 = kX_1, \quad X_2 = X_3 = X_4 = 0$.

It follows, therefore, that to the points infinitely near the point O_{1333} correspond projectively on the surface F_1 the points on the straight line

$$(a_1) \qquad X_0 = X_2 = X_3 = X_4 = 0$$

This also means that the points infinitely near O_{1333} correspond to the points infinitely near the point O_0' on F and lying in the plane

$$X_2 = X_3 = X_4 = 0 \ .$$

This plane is tangent to F at the point O_0'.

We have now established the fact that to the invariant point O_1 of the involution I_7, corresponds on the surface F a branch point O_0' , which is a four-tuple point. The fourth order tangent cone at this point has degenerated into a cubic tangent cone

$$(9) \qquad \begin{aligned} X_1X_3 &= X_2^2 \\ X_2X_4 &= X_3^2 \\ X_5 &= 0 \end{aligned}$$

and a tangent plane

$$(10) \qquad X_2 = X_3 = X_4 = 0$$

Moreover, the cubic curve γ_1 and the straight line a_1 have in common only the point

$$X_0 = X_2 = X_3 = X_4 = X_5 = 0,$$

which is designated by $O_1'(0,1,0,0,0,0)$.

The cone (9) and the plane (10) have in common only the straight line

$$X_2 = X_3 = X_4 = X_5 = 0 \ .$$

Hence, the tangent plane is also tangent to the cubic cone along this triple line.

Images of Curves $|C_1|$ at O_0'.

Turning now to the study of the system of curves $|C_1|$, one has

$$u_0 x_1^4 x_3^3 + u_1 x_1^3 x_2^2 x_3^2 + u_2 x_1^2 x_2^4 x_3 + u_3 x_1 x_2^6 + u_4 x_2 x_3^6 = 0$$

These curves pass through the invariant point O_1, which is triple for these curves. Each branch is tangent to the invariant direction $x_3 = 0$.

Applying the transformation U to the curves $|C_1|$ gives

$$u_0 z_1^7 z_3^3 + u_1 z_1^7 z_2 z_3^2 + u_2 z_1^7 z_2^2 z_3 + u_3 z_1^7 z_2^3 + u_4 z_2^4 z_3^6 = 0$$

or

$$(11) \quad z_1^7 (u_0 z_3^3 + u_1 z_2 z_3^2 + u_2 z_2^2 z_3 + u_3 z_2^3) + u_4 z_2^4 z_3^6 = 0$$

These curves have a triple point at the point ($z_2 = z_3 = 0$), and the tangents to the curves at this point have the equations

$$u_0 z_3^3 + u_1 z_2 z_3^2 + u_2 z_2^2 z_3 + u_3 z_2^3 = 0.$$

Thus, the point O_{12}, in the first order neighborhood of O_1, is triple. Hence, there are three simple

variable points in the first order neighborhood of O_{12}.

To approach the point ($z_2 = z_3 = 0$) along the curves (11), one substitutes $z_3 = kz_2$ into their equation and allows z_2 to approach zero.

It follows that

$$z_1^7(u_0k^3z_2^3 + u_1k^2z_2^3 + u_2kz_2^3 + u_3z_2^3) + u_4k^6z_2^{10} = 0$$

or

$$z_1^7(u_0k^3 + u_1k^2 + u_2k + u_3\ ) + u_4k^6z_2^7 = 0,$$

and hence

(12) $$u_0k^3 + u_1k^2 + u_2k + u_3 = 0$$

One has learned earlier that the points in the first order neighborhood of O_{12} project into the points of the twisted cubic curve Υ_1. Any one member of the system $|C_1|$ has three points in the first order neighborhood of O_{12}, and their projections on F_1 are therefore on the twisted cubic curve Υ_1. The three values for k, found by solving equation (12), will therefore give the values for k in the parametric equations (6), which will locate the three points on Υ_1.

Assume that the roots of (12) are k', k'', and k'''. The three points then have the coordinates

(Q_1) $(0, k'^3, k'^2, k', 1, 0)$

(Q_2) $(0, k''^3, k''^2, k'', 1, 0)$

(Q_3) $(0, k'''^3, k'''^2, k''', 1, 0)$

These three points determine the plane

$$X_1 + aX_2 + bX_3 + cX_4 = 0, \qquad X_0 = X_5 = 0$$

whose coefficients will be evaluated below. Substituting the roots in the equation for the plane gives

$$k'^3 + ak'^2 + bk' + c = 0$$

$$k''^3 + ak''^2 + bk'' + c = 0$$

$$k'''^3 + ak'''^2 + bk''' + c = 0.$$

When these equations are solved simultaneously for a, b, and c, one gets

$$a = -(k' + k'' + k''')$$

$$b = k'k'' + k'k''' + k''k'''$$

$$c = -k'k''k'''$$

By making use of the relationship between roots and coefficients of rational integral equations, it is evident from (12)

$$u_0k^3 + u_1k^2 + u_2k + u_3 = 0$$

or $$k^3 + \frac{u_1}{u_0}k^2 + \frac{u_2}{u_0}k + \frac{u_3}{u_0} = 0,$$

that $\dfrac{u_1}{u_0} = -(k' + k'' + k''') = a$

and $\frac{u_2}{u_0} = k'k'' + k'k''' + k''k''' = b$

and $\frac{u_3}{u_0} = -k'k''k''' = c$

So the equation of the plane, $Q_1Q_2Q_3$, can now be written as

$$X_1 + \frac{u_1}{u_0} X_2 + \frac{u_2}{u_0} X_3 + \frac{u_3}{u_0} X_4 = 0, \qquad X_0 = X_5 = 0$$

or

$$u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4 = 0, \qquad X_0 = X_5 = 0$$

These results show that the images, designated by $\left|\Gamma_1\right|$, of the curves $\left|C_1\right|$ mapped upon the surface F, have a triple point at O_0', and the tangents to the curves at this point are the intersections of the third order tangent cone (9) with the hyperplane

$$u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4 = 0$$

When the curves $\left|\Gamma_1\right|$ are projected from O_0' upon the cubic surface F_1, the equations for the curves, now designated by $\left|\Gamma_1\right|$, become

$$\left\|\begin{matrix} X_1 & X_2 & X_3 & (-u_4X_5) \\ X_2 & X_3 & X_4 & (u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4) \end{matrix}\right\| = 0$$

$$X_0 = 0$$

Using again the method illustrated by F. S. Woods [15], and applied earlier in this chapter to the surface F, the order of the curves $|\Gamma_1'|$ is four. Since one more equation is required to properly express a curve than is necessary for a surface, only one cutting hyperplane is used.

<u>Images of Curves</u> $|C_2|$ <u>at</u> O_0'.

One turns now to the study of the curves $|C_2|$. Their equation is

$$u_0x_1^4x_2x_3^2 + u_1x_1^3x_2^3x_3 + u_2x_1^2x_2^5 + u_3x_1x_3^6 + u_4x_2^2x_3^5 = 0$$

They have a triple point at O_1, with tangents $x_2 = 0$ and $x_3 = 0$ (the latter counted twice).

Applying the transformation U to this equation gives

$$u_0z_1^7z_3^2 + u_1z_1^7z_2z_3 + u_2z_1^7z_2^2 + u_3z_2^3z_3^6 + u_4z_2^4z_3^5 = 0$$

or

$$(13)\quad z_1^7(u_0z_3^2 + u_1z_2z_3 + u_2z_2^2) + u_3z_2^3z_3^6 + u_4z_2^4z_3^5 = 0$$

These curves have a double point at the point ($z_2 = z_3 = 0$), and the tangents to the curves at this point have the equation

$$u_0z_3^2 + u_1z_2z_3 + u_2z_2^2 = 0 .$$

Thus the point O_{12}, in the first order neighborhood of

O_1, is a double point, and there are two simple variable points in the first order neighborhood of O_{12}.

To approach the point $(z_2 = z_3 = 0)$ along the curves (13), substitute $z_3 = kz_2$ and allow z_2 to approach zero. The results are

$$z_1^7(u_0k^2z_2^2 + u_1kz_2^2 + u_2z_2^2) + u_3k^6z_2^9 + u_4k^5z_2^9 = 0$$

or

$$z_1^7(u_0k^2 + u_1k + u_2) + u_3k^6z_2^7 + u_4k^5z_2^7 = 0 \, .$$

Hence,

$$u_0k^2 + u_1k + u_2 = 0 \, .$$

The coordinates, then, of the two points on Υ_1, corresponding to the two variable points, infinitely near O_{12}, are therefore

$$(0, k'^3, k'^2, k', 1, 0)$$

and

$$(0, k''^3, k''^2, k'', 1, 0)$$

where k' and k'' are the roots of

$$u_0k^2 + u_1k + u_2 = 0$$

Applying, now, the transformation V, successively three times, to the curves $|C_2|$, gives

$$u_0z_1^6z_2 + u_1z_1^3z_2^3z_3 + u_2z_2^5z_3^2 + u_3z_1^4z_3^3 + u_4z_1z_2^2z_3^4 = 0$$

and

$$u_0 z_1^{10} z_2 + u_1 z_1^5 z_2^3 z_3^3 + u_2 z_2^5 z_3^6 + u_3 z_1^9 z_3^2 + u_4 z_1^4 z_2^2 z_3^5 = 0$$

and

$$u_0 z_1^{14} z_2 + u_1 z_1^7 z_2^3 z_3^5 + u_2 z_2^5 z_3^{10} + u_3 z_1^{14} z_3 + u_4 z_1^7 z_2^2 z_3^6 = 0$$

or

$$(14) \quad z_1^{14}(u_0 z_2 + u_3 z_3) + u_1 z_1^7 z_2^3 z_3^5 + u_2 z_2^5 z_3^{10} + u_4 z_1^7 z_2^2 z_3^6 = 0$$

This shows that the curves $|C_2|$ pass simply through the points O_{13}, O_{133}, and O_{1333} in the first, second, and third order neighborhoods of O_1. They have a simple variable point infinitely near O_{1333}.

To find the image of this point on F_1, one again substitutes $z_3 = kz_2$ into the equation (14) and gets

$$z_1^{14}(u_0 z_2 + u_3 k z_2) + u_1 z_1^7 k^5 z_2^8 + u_2 k^{10} z_2^{15} + u_4 z_1^7 k^6 z_2^8 = 0$$

or

$$z_1^{14}(u_0 + u_3 k) + u_1 z_1^7 k^5 z_2^7 + u_2 k^{10} z_2^{14} + u_4 k^6 z_1^7 z_2^7 = 0$$

By allowing z_2 to approach zero, one gets

$$u_0 + u_3 k = 0$$

or

$$k = \frac{-u_0}{u_3} .$$

This is the value of k to be used in equation (8), when one locates the image point on the straight line

$$X_0 = X_2 = X_3 = X_4 = 0 .$$

Hence, the coordinates of this point become

$$(0,\ 1,\ 0,\ 0,\ 0,\ \frac{-u_0}{u_3})$$

or $$(0,\ u_3,\ 0,\ 0,\ 0,\ -u_0)$$

These results then show that the curves $\left|\Gamma_2\right|$, on the surface F, have a triple point at O_0'. Two of the tangent lines at this point lie on the tangent cone (9) and have the equations

$$\frac{X_1}{k^3} = \frac{X_2}{k^2} = \frac{X_3}{k} = \frac{X_4}{1}, \qquad X_5 = 0$$

where the values of k are the roots of

$$u_0k^2 + u_1k + u_2 = 0 .$$

The third tangent line lies in the tangent plane (10) and has for equations

$$u_0X_1 + u_3X_5 = 0, \qquad X_2 = X_3 = X_4 = 0 .$$

<u>Images of Curves $\left|C_3\right|$ at O_0'.</u>

One continues now with the curves $\left|C_3\right|$, whose equation is

$$u_0x_1^5x_3^2 + u_1x_1^4x_2^2x_3 + u_2x_1^3x_2^4 + u_3x_1x_2x_3^5 + u_4x_2^3x_3^4 = 0$$

Applying the transformation U gives

$$u_0z_1^7z_3^2 + u_1z_1^7z_2z_3 + u_2z_1^7z_2^2 + u_3z_2^4z_3^5 + u_4z_2^5z_3^4 = 0$$

or

$$z_1^7(u_0z_3^2 + u_1z_2z_3 + u_2z_2^2) + u_3z_2^4z_3^5 + u_4z_2^5z_3^4 = 0 \tag{15}$$

These curves have, therefore, a double point at the point ($z_2 = z_3 = 0$), and the tangents to the curves at this point have the equation

$$u_0z_3^2 + u_1z_2z_3 + u_2z_2^2 = 0$$

Thus the point O_{12} is double, and there are two simple variable points in the first order neighborhood of the point O_{12}.

To approach the point ($z_2 = z_3 = 0$) along the curves (15), substitute $z_3 = kz_2$ into their equation and allow z_2 to approach zero. The results are

$$z_1^7(u_0k^2z_2^2 + u_1kz_2^2 + u_2z_2^2) + u_3k^5z_2^9 + u_4k^4z_2^9 = 0$$

or

$$z_1^7(u_0k^2 + u_1k + u_2) + u_3k^5z_2^7 + u_4k^4z_2^7 = 0$$

and $$u_0k^2 + u_1k + u_2 = 0$$

If the roots of this last equation are k' and k'', then the two points on Υ_1, corresponding to the two variable points in the first order neighborhood of O_{12}, have for coordinates

$$(0,\ k'^3,\ k'^2,\ k',\ 1,\ 0)$$

and

$$(0,\ k''^3,\ k''^2,\ k'',\ 1,\ 0)$$

The curves $|\Gamma_3|$, the images of the curves $|C_3|$ on F, have therefore a double point at O_0', and the two tangent lines to the curves at this point lie on the cone (9), and have for equations

$$\frac{x_1}{k^3} = \frac{x_2}{k^2} = \frac{x_3}{k} = \frac{x_4}{1}, \qquad x_5 = 0$$

where the values of k are the roots of

$$u_0 k^2 + u_1 k + u_2 = 0 .$$

<u>Images of Curves $|C_4|$ at O_0'.</u>

One takes up now the study of the curves $|C_4|$; they have the equation

$$u_0 x_1^5 x_2 x_3 + u_1 x_1^4 x_2^3 + u_2 x_1^2 x_3^5 + u_3 x_1 x_2^2 x_3^4 + u_4 x_2^4 x_3^3 = 0$$

Applying the transformation U to these curves, gives

$$u_0 z_1^7 z_3 + u_1 z_1^7 z_2 + u_2 z_2^3 z_3^5 + u_3 z_2^4 z_3^4 + u_4 z_2^5 z_3^3 = 0$$

or

(16) $$z_1^7(u_0 z_3 + u_1 z_2) + u_2 z_2^3 z_3^5 + u_3 z_2^4 z_3^4 + u_4 z_2^5 z_3^3 = 0$$

Thus, the curves (16) have a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to these curves at

this point has for its equation

$$u_0z_3 + u_1z_2 = 0$$

The point O_{12} is therefore simple, and there is one simple variable point in the first order neighborhood of O_{12}.

To approach the point $(z_2 = z_3 = 0)$ along the curves (16), substitute $z_3 = kz_2$ into their equation and allow z_2 to approach zero. The results are

$$z_1^7(u_0kz_2 + u_1z_2) + u_2k^5z_2^8 + u_3k^4z_2^8 + u_4k^3z_2^8 = 0$$

$$z_1^7(u_0k + u_1) + u_2k^5z_2^7 + u_3k^4z_2^7 + u_4k^3z_2^7 = 0$$

$$u_0k + u_1 = 0$$

The value of k from this equation, substituted into the equation (6), locates the point on Υ_1, corresponding to the point on $|C_4|$, in the first order neighborhood of O_{12}. The coordinates of this point on Υ_1 are

$$(0,\ -u_1^3,\ u_0u_1^2,\ -u_0^2u_1,\ u_0^3,\ 0).$$

Applying now the transformation V successively three times to the curves $|C_4|$ gives

$$u_0z_1^8z_2 + u_1z_1^5z_2^3z_3 + u_2z_1^6z_3^3 + u_3z_1^3z_2^2z_3^4 + u_4z_2^4z_3^5 = 0$$

and $$u_0z_1^{11}z_2 + u_1z_1^6z_2^3z_3^3 + u_2z_1^{10}z_3^2 + u_3z_1^5z_2^2z_3^5 + u_4z_2^4z_3^8 = 0$$

and $$u_0z_1^{14}z_2 + u_1z_1^7z_2^3z_3^5 + u_2z_1^{14}z_3 + u_3z_1^7z_2^2z_3^6 + u_4z_2^4z_3^{11} = 0$$

or

$$(17) \qquad z_1^{14}(u_0z_2 + u_2z_3) + u_1z_1^7z_2^3z_3^5 + u_3z_1^7z_2^2z_3^6 + u_4z_2^4z_3^{11} = 0$$

The curves (17) have a simple point at $(z_2 = z_3 = 0)$, and the variable tangent to the curves at this point has for its equation

$$u_0z_2 + u_2z_3 = 0$$

This further shows that the curves $|C_4|$ pass simply through the points O_{13}, O_{133}, and O_{1333}, and have a simple variable point in the first order neighborhood of the point O_{1333}.

In order now to approach the point $(z_2 = z_3 = 0)$ along the curves (17), substitute into these equations $z_3 = kz_2$, and allow z_2 to approach zero. This gives

$$z_1^{14}(u_0z_2 + u_2kz_2) + u_1z_1^7k^5z_2^8 + u_3z_1^7k^6z_2^8 + u_4k^{11}z_2^{15} = 0$$

or $z_1^{14}(u_0 + u_2k) + u_1z_1^7k^5z_2^7 + u_3z_1^7k^6z_2^7 + u_4k^{11}z_2^{14} = 0.$

and $$u_0 + u_2k = 0 \ .$$

The value of k from this equation, substituted into equation (8), locates the point on the line $X_0 = X_2 = X_3 = X_4 = 0$, corresponding to the point of $|C_4|$ infinitely near O_{1333}.

The curves $|\Gamma_4|$ have then a double point at O_0',

and the equations of the tangent lines are

$$\frac{x_1}{-u_1^3} = \frac{x_2}{u_0u_1^2} = \frac{x_3}{-u_0^2u_1} = \frac{x_4}{u_0^3}, \quad x_5 = 0$$

and $$u_0x_1 + u_2x_5 = 0, \qquad x_2 = x_3 = x_4 = 0$$

The first of these tangents lies on the cone (9), while the second lies on the plane (10).

Images of Curves $|C_5|$ at O_0'.

The curves $|C_5|$ will be studied next. They have for their equation

$$u_0x_1^6x_3 + u_1x_1^5x_2^2 + u_2x_1^2x_2x_3^4 + u_3x_1x_2^3x_3^3 + u_4x_2^5x_3^2 = 0$$

Applying the transformation U gives

$$u_0z_1^7z_3 + u_1z_1^7z_2 + u_2z_2^4z_3^4 + u_3z_2^5z_3^3 + u_4z_2^6z_3^2 = 0$$

or

$$(18) \quad z_1^7(u_0z_3 + u_1z_2) + u_2z_2^4z_3^4 + u_3z_2^5z_3^3 + u_4z_2^6z_3^2 = 0$$

Thus, the curves $|C_5|$ have a simple point at $z_2 = z_3 = 0$. The equation of the variable tangent is

$$u_0z_3 + u_1z_2 = 0 .$$

O_{12} is a simple point for $|C_5|$, and there is a simple variable point in the first order neighborhood of O_{12}.

Substituting $z_3 = kz_2$ in (18), and allowing z_2 to approach zero, will give

$$z_1^7(u_0 k z_2 + u_1 z_2) + u_2 k^4 z_2^8 + u_3 k^3 z_2^8 + u_4 k^2 z_2^8 = 0$$

or $z_1^7(u_0 k + u_1) + u_2 k^4 z_2^7 + u_3 k^3 z_2^7 + u_4 k^2 z_2^7 = 0$.

Thus $$u_0 k + u_1 = 0 \ .$$

The value of k, substituted in (6), will locate on Υ_1 the image point of the point in the first order neighborhood of O_{12}. The coordinates of this point are

$$(0, -u_1^3, u_0 u_1^2, -u_0^2 u_1, u_0^3, 0) \ .$$

The curves $|\Gamma_5|$ have therefore a simple point at O_0', and the equations of their tangent line, lying on cone (9), are

$$\frac{x_1}{-u_1^3} = \frac{x_2}{u_0 u_1^2} = \frac{x_3}{-u_0^2 u_1} = \frac{x_4}{u_0^3}\ , \quad x_5 = 0 \ .$$

<u>Images of Curves $|C_6|$ at O_0'.</u>

Finally, one studies the curves $|C_6|$, which are

$$u_0 x_1^6 x_2 + u_1 x_1^3 x_3^4 + u_2 x_1^2 x_2^2 x_3^3 + u_3 x_1 x_2^4 x_3^2 + u_4 x_2^6 x_3 = 0$$

Applying the transformation V three times in succession gives

$$u_0 z_1^{11} z_2 + u_1 z_1^9 z_3^3 + u_2 z_1^6 z_2^2 z_3^4 + u_3 z_1^3 z_2^4 z_3^5 + u_4 z_2^6 z_3^6 = 0$$

and $$u_0 z_1^{16} z_2 + u_1 z_1^{15} z_3^2 + u_2 z_1^{10} z_2^2 z_3^5 + u_3 z_1^5 z_2^4 z_3^8 + u_4 z_2^6 z_3^{11} = 0$$

and $u_0z_1^{21}z_2 + u_1z_1^{21}z_3 + u_2z_1^{14}z_2^2z_3^6 + u_3z_1^7z_2^4z_3^{11} + u_4z_2^6z_3^{16} = 0$

or

(19) $z_1^{21}(u_0z_2 + u_1z_3) + u_2z_1^{14}z_2^2z_3^6 + u_3z_1^7z_2^4z_3^{11} + u_4z_2^6z_3^{16} = 0$

Thus, the curves (19) have a simple point at ($z_2 = z_3 = 0$), with variable tangent

$$u_0z_2 + u_1z_3 = 0 .$$

The curves $|C_6|$ pass simply through the points O_{13}, O_{133}, and O_{1333}. They have a simple variable point in the first order neighborhood of O_{1333}.

One, next, substitutes $z_3 = kz_2$ in (19) and allows z_2 to approach zero. This gives

$$z_1^{21}(u_0z_2 + u_1kz_2) + u_2z_1^{14}k^6z_2^8 + u_3z_1^7k^{11}z_2^{15} + u_4k^{16}z_2^{22} = 0$$

or $z_1^{21}(u_0 + u_1k) + u_2z_1^{14}k^6z_2^7 + u_3z_1^7k^{11}z_2^{14} + u_4k^{16}z_2^{21} = 0$

Hence, $$u_0 + u_1k = 0 .$$

When the value of k in this equation is substituted in (8), one gets the point on the line

$$X_0 = X_2 = X_3 = X_4 = 0,$$

corresponding to the point of $|C_6|$ infinitely near O_{1333}. The coordinates of this point are

$$(0, u_1, 0, 0, 0, -u_0)$$

The curves $|\Gamma_6|$ have, therefore, a simple point on F at O_0', and the equations for its tangent line at this point, lying in the plane (10), are

$$u_0X_1 + u_1X_5 = 0, \qquad X_2 = X_3 = X_4 = 0 .$$

<u>Branch</u> <u>Point</u> <u>O_5'</u>.

As has already been pointed out in the beginning of this chapter, there exists a harmonic homology, which not only transforms the involution plane I_7 into itself, but also its image surface F. The invariant point O_1 goes into O_3, and the branch point O_0' goes into $O_5'(0,0,0,0,0,1)$. The points $O_2(0,1,0)$, $O_1'(0,1,0,0,0,0)$, $O_2'(0,0,1,0,0,0)$, $O_3'(0,0,0,1,0,0)$, and $O_4'(0,0,0,0,1,0)$ go into themselves. In addition, it was also seen that

$$\begin{array}{ll}
|C_1| \longrightarrow |C_6| & \qquad |C_4| \longrightarrow |C_3| \\
|C_2| \longrightarrow |C_5| & \qquad |C_5| \longrightarrow |C_2| \\
|C_3| \longrightarrow |C_4| & \qquad |C_6| \longrightarrow |C_1| \\
|\Gamma_1| \longrightarrow |\Gamma_6| & \qquad |\Gamma_4| \longrightarrow |\Gamma_3| \\
|\Gamma_2| \longrightarrow |\Gamma_5| & \qquad |\Gamma_5| \longrightarrow |\Gamma_2| \\
|\Gamma_3| \longrightarrow |\Gamma_4| & \qquad |\Gamma_6| \longrightarrow |\Gamma_1|
\end{array}$$

One can, therefore, at once write down the characteristics of the branch point O_5' for the surface F and

the curves $\left|\Gamma_i\right|$ ($i = 1, 2, 3, 4, 5, 6$) which pass through this point.

To the invariant point O_3, corresponds, then, on the surface F a branch point O_5'. It is a four-tuple point for the surface F. The quartic tangent cone has degenerated into a cubic cone

(20) $$x_1x_3 = x_2^2$$
$$x_2x_4 = x_3^2$$
$$x_0 = 0$$

and a plane

(21) $$x_2 = x_3 = x_4 = 0 .$$

This cone and plane have in common the straight line

$$x_0 = x_2 = x_3 = x_4 = 0$$

The curves $\left|\Gamma_6\right|$ have a triple point at O_5', and the tangents to the curves at this point are the intersections of the cubic tangent cone (20) with the hyperplane

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0.$$

The curves $\left|\Gamma_5\right|$ have a triple point at O_5', and the tangents to the curves at this point consist of two tangents lying in (20), and the third tangent lying in the tangent

plane (21).

The equations of the tangent lines are respectively

$$\frac{x_1}{k^3} = \frac{x_2}{k^2} = \frac{x_3}{k} = \frac{x_4}{1}, \qquad x_0 = 0,$$

where the values of k are the roots of

$$u_2k^2 + u_3k + u_4 = 0,$$

and $$u_2x_1 + u_0x_0 = 0 \qquad x_2 = x_3 = x_4 = 0$$

The curves $|\Gamma_4|$ have a double point at O_5', and the equations of the two tangents at this point, lying on the cubic cone (20), are

$$\frac{x_1}{k^3} = \frac{x_2}{k^2} = \frac{x_3}{k} = \frac{x_4}{1} \qquad x_0 = 0$$

where the values of k are the roots of

$$u_2k^2 + u_3k + u_4 = 0 .$$

The curves $|\Gamma_3|$ have a double point at O_5', and the equations of the tangent lines are

$$\frac{x_1}{-u_4^3} = \frac{x_2}{u_3u_4^2} = \frac{x_3}{-u_3^2u_4} = \frac{x_4}{u_3^3} \qquad x_0 = 0$$

and $$u_0x_0 + u_3x_1 = 0, \qquad x_2 = x_3 = x_4 = 0$$

The first of these tangents lies on the cubic cone (20), while the second lies on the plane (21).

The curves $|\Gamma_2|$ have a simple point at O_5', and the

equations of the tangent line through this point, and lying on the cubic cone (20), are

$$\frac{x_1}{-u_4^3} = \frac{x_2}{u_3u_4^2} = \frac{x_3}{-u_3^2u_4} = \frac{x_4}{u_3^3} \qquad x_0 = 0.$$

Finally, the curves $|\Gamma_1|$ have a simple point on F at O_5', and the equations of their tangent line at this point, and lying in the plane (21), are

$$u_0x_0 + u_4x_1 = 0, \qquad x_2 = x_3 = x_4 = 0 \; .$$

<u>Branch Point O_4' and Adjacent Points O_3', O_2'.</u>

There remains, finally, to be studied the singularity of the branch point $O_4'(x_0 = x_1 = x_2 = x_3 = x_5 = 0)$, which corresponds to the point O_2 of the involution I_7.

When the surface F is projected onto the hyperplane $x_4 = 0$ from O_4', one obtains the fifth order surface

$$F_2, \qquad \begin{aligned} x_1x_3 &= x_2^2 \\ x_1^2x_2 &= x_0x_3x_5 \\ x_4 &= 0 \end{aligned}$$

Projecting also F_2 from the point O_3' onto the hyperplane $x_3 = 0$, gives the cubic surface

$$F_3, \qquad \begin{aligned} x_1^3 &= x_0x_2x_5 \\ x_3 &= 0 \\ x_4 &= 0 \; . \end{aligned}$$

Finally, projecting F_3 from the point O_2' onto the hyperplane $X_2 = 0$, gives the plane

$$F_4, \qquad X_2 = X_3 = X_4 = 0$$

Consider now the system of curves $|C_0|$, passing through the point O_2. They are

$$(22) \quad v_3x_2^5x_1x_3 + v_2x_2^3x_1^2x_3^2 + v_1x_2x_1^3x_3^3 + v_0x_1^7 + v_5x_3^7 = 0.$$

Applying the transformation

$$W, \qquad x_1 : x_2 : x_3 = z_1z_3 : z_2^2 : z_2z_3$$

five times, in succession, gives

$$v_3z_2^{11}z_1 + v_2z_2^8z_1^2z_3^2 + v_1z_2^5z_1^3z_3^4 + v_0z_1^7z_3^5 + v_5z_2^7z_3^5 = 0$$

and

$$v_3z_2^{17}z_1 + v_2z_2^{13}z_1^2z_3^3 + v_1z_2^9z_1^3z_3^6 + v_0z_1^7z_3^{11} + v_5z_2^{14}z_3^4 = 0$$

and

$$v_3z_2^{23}z_1 + v_2z_2^{18}z_1^2z_3^4 + v_1z_2^{13}z_1^3z_3^8 + v_0z_1^7z_3^{17} + v_5z_2^{21}z_3^3 = 0$$

and

$$v_3z_2^{29}z_1 + v_2z_2^{23}z_1^2z_3^5 + v_1z_2^{17}z_1^3z_3^{10} + v_0z_1^7z_3^{23} + v_5z_2^{28}z_3^2 = 0$$

and

$$(23) \quad z_2^{35}(v_3z_1 + v_5z_3) + v_2z_2^{28}z_1^2z_3^6 + v_1z_2^{21}z_1^3z_3^{12} + v_0z_1^7z_3^{29} = 0.$$

The curves (23), therefore, have a simple point at $(z_1 = z_3 = 0)$, with a variable tangent

$$v_3z_1 + v_5z_3 = 0 .$$

The curves (22) pass simply through the first five neighborhoods of O_2 on the invariant line $x_1 = 0$. The points will be designated by O_{23}, O_{233}, O_{2333}, O_{23333}, and O_{233333}. There is a simple variable point in the first order neighborhood of O_{233333}.

Applying also the transformation W five times to

$$\frac{x_3}{x_2^5 x_1 x_3} = \frac{x_2}{x_2^3 x_1^2 x_3^2} = \frac{x_1}{x_2 x_1^3 x_3^3} = \frac{x_0}{x_1^7} = \frac{x_5}{x_3^7},$$

gives

$$\frac{x_3}{z_2^{35} z_1} = \frac{x_2}{z_2^{28} z_1^2 z_3^6} = \frac{x_1}{z_2^{21} z_1^3 z_3^{12}} = \frac{x_0}{z_1^7 z_3^{29}} = \frac{x_5}{z_2^{35} z_3}.$$

Making the substitution $z_3 = kz_1$, gives

$$\frac{x_3}{z_2^{35} z_1} = \frac{x_2}{k^6 z_2^{28} z_1^8} = \frac{x_1}{k^{12} z_2^{21} z_1^{15}} = \frac{x_0}{k^{29} z_1^{36}} = \frac{x_5}{k z_2^{35} z_1}$$

or

$$\frac{x_3}{z_2^{35}} = \frac{x_2}{k^6 z_2^{28} z_1^7} = \frac{x_1}{k^{12} z_2^{21} z_1^{14}} = \frac{x_0}{k^{29} z_1^{35}} = \frac{x_5}{k z_2^{35}}$$

Allowing z_1 to approach zero, gives

(24) $$\frac{x_3}{1} = \frac{x_5}{k}, \qquad x_0 = x_1 = x_2 = 0$$

Thus, the points infinitely near O_{233333} correspond

projectively to the points common to the surface F_2 and the plane

$$x_0 = x_1 = x_2 = 0 ,$$

which is tangent to F at the point O_4'. It follows that the points in the first order neighborhood of O_{233333} correspond projectively, on the surface F_2, to the points of the straight line

$$a_{233333}, \quad x_0 = x_1 = x_2 = x_4 = 0.$$

Since there is a harmonic homology, and since both surfaces F and F_2, as well as the points O_4' and O_3', are invariant, one can immediately write down the results for the second branch of the curves (22).

The curves (23) have a simple point at $(z_1 = z_3 = 0)$, with variable tangent

$$v_3 z_3 + v_0 z_1 = 0 .$$

The curves (22) pass simply through the first five neighborhoods of O_2 on the invariant line $x_3 = 0$. The points will be designated by O_{21}, O_{211}, O_{2111}, O_{21111}, and O_{211111}; and there is a simple variable point in the first order neighborhood of O_{211111}.

To the points infinitely near O_{211111} correspond projectively on the surface F_2 the points of the straight line

$$a_{211111}, \quad x_5 = x_1 = x_2 = x_4 = 0 .$$

Hence, one concludes that the surface F has a double bi-planar point at O_4', the tangent planes being

$$X_0 = X_1 = X_2 = 0$$

$$X_5 = X_1 = X_2 = 0$$

Consider next, the system of curves

$$v_2x_2^3x_1^2x_3^2 + v_1x_2x_1^3x_3^3 + v_0x_1^7 + v_5x_3^7 = 0 \tag{25}$$

These curves have at O_2 a four-tuple point. Two of the tangents are $x_1 = 0$, (counted twice), while the other two are $x_3 = 0$, (also counted twice).

Applying twice, the transformation W gives

$$v_2z_2^8z_1^2 + v_1z_2^5z_1^3z_3^2 + v_0z_1^7z_3^3 + v_5z_2^7z_3^3 = 0$$

and

$$v_2z_2^{13}z_1^2 + v_1z_2^9z_1^3z_3^3 + v_0z_1^7z_3^8 + v_5z_2^{14}z_3 = 0.$$

Applying next, the transformation

$$R, \quad x_1 : x_2 : x_3 = z_1z_2 : z_2^2 : z_1z_3$$

gives

$$v_2z_2^{21}z_1 + v_1z_2^{14}z_1^5z_3^3 + v_0z_1^{14}z_3^8 + v_5z_2^{21}z_3 = 0$$

or

$$z_2^{21}(v_2z_1 + v_5z_3) + v_1z_2^{14}z_1^5z_3^3 + v_0z_1^{14}z_3^8 = 0. \tag{26}$$

Thus, the curves (26) have a simple point at

$(z_1 = z_3 = 0)$, with variable tangent

$$v_2 z_1 + v_5 z_3 = 0 .$$

The curves (25) have a double point at O_{23}, a simple point at O_{233}, a simple point at O_{2331}, and a simple variable point in the first order neighborhood of O_{2331}.

Next, let

$$\frac{X_2}{x_2^3 x_1^2 x_3^2} = \frac{X_1}{x_2 x_1^3 x_3^3} = \frac{X_0}{x_1^7} = \frac{X_5}{x_3^7} .$$

After applying the transformations W, twice, and R, once, one gets

$$\frac{X_2}{z_2^{21} z_1} = \frac{X_1}{z_2^{14} z_1^5 z_3^3} = \frac{X_0}{z_1^{14} z_3^8} = \frac{X_5}{z_2^{21} z_3} .$$

If $z_3 = kz_1$, and z_1 approaches zero, one obtains

$$\frac{X_2}{z_2^{21} z_1} = \frac{X_1}{k^3 z_2^{14} z_1^8} = \frac{X_0}{k^8 z_1^{22}} = \frac{X_5}{kz_2^{21} z_1}$$

or

$$\frac{X_2}{z_2^{21}} = \frac{X_1}{k^3 z_2^{14} z_1^7} = \frac{X_0}{k^8 z_1^{21}} = \frac{X_5}{kz_2^{21}} ,$$

and hence,

(27) $$\frac{X_2}{1} = \frac{X_5}{k} \qquad X_1 = X_0 = 0$$

Therefore, it follows that to the points infinitely near O_{2331} correspond projectively the points on the surface F_3,

and lying on the straight line,

$$b_{2331}, \qquad x_0 = x_1 = x_3 = x_4 = 0 .$$

Because of the existing harmonic homology, one can immediately say that the curves (25) have a double point at O_{21}, a simple point at O_{211}, a simple point at O_{2113}, and a simple variable point in the first order neighborhood of O_{2113}. Hence, to the points in this latter neighborhood correspond, projectively, the points on the surface F_3, and lying on the straight line

$$b_{2113}, \qquad x_5 = x_1 = x_3 = x_4 = 0 .$$

Consider next, the system of curves

$$(28) \qquad v_1x_2x_1^3x_3^3 + v_0x_1^7 + v_5x_3^7 = 0 .$$

Applying the transformation W, once, gives

$$v_1z_2^5z_1^3 + v_0z_1^7z_3 + v_5z_2^7z_3 = 0 .$$

Applying the transformation R, twice, gives

$$v_1z_2^6z_1^2 + v_0z_1^7z_3 + v_5z_2^7z_3 = 0$$

and

$$v_1z_2^7z_1 + v_0z_1^7z_3 + v_5z_2^7z_3 = 0$$

or

$$(29) \qquad z_2^7(v_1z_1 + v_5z_3) + v_0z_1^7z_3 = 0$$

Thus, the curves (29) have a simple point at $(z_1 = z_3 = 0)$, with the variable tangent

$$v_1 z_1 + v_5 z_3 = 0 \ .$$

Furthermore, the curves (28) pass simply through O_{23}, O_{231}, O_{2311}, and have a simple variable point in the first order neighborhood of O_{2311}.

When

$$\frac{x_1}{x_2 x_1^3 x_3^3} = \frac{x_0}{x_1^7} = \frac{x_5}{x_3^7} \ ,$$

and when the transformations W and T are applied once and twice, respectively, it follows that

$$\frac{x_1}{z_2^7 z_1} = \frac{x_0}{z_1^7 z_3} = \frac{x_5}{z_2^7 z_3} \ .$$

Again use $z_3 = kz_1$ and let z_1 approach zero. This gives

$$\frac{x_1}{z_2^7 z_1} = \frac{x_0}{kz_1^8} = \frac{x_5}{kz_2^7 z_1}$$

or

$$\frac{x_1}{z_2^7} = \frac{x_0}{kz_1^7} = \frac{x_5}{kz_2^7}$$

and

(30) $$\frac{x_1}{1} = \frac{x_5}{k} \qquad x_0 = 0$$

Thus, to the points infinitely near O_{2311} correspond, projectively, on the plane F_4, the points lying on the straight line

$$c_{2311}, \qquad X_0 = X_2 = X_3 = X_4 = 0 \; .$$

By the harmonic homology one can also say that the curves (28) pass simply through the points O_{21}, O_{213}, O_{2133}, and have a simple variable point in the first order neighborhood of O_{2133}.

To the points in this latter neighborhood correspond, projectively, the points on the plane F_4, lying on the line

$$c_{2133}, \qquad X_5 = X_2 = X_3 = X_4 = 0 \; .$$

Finally, one has the equation

$$v_0 x_1^7 + v_5 x_3^7 = 0 \; . \tag{31}$$

These curves degenerate into seven straight lines all passing through the point O_2. To one of these degenerate curves corresponds, projectively, on the plane F_4 the straight line

$$v_0 X_0 + v_5 X_5 = 0 \; , \qquad X_2 = X_3 = X_4 = 0 \, .$$

Therefore, it follows that to points infinitely near O_2, on the seven straight lines considered, correspond a point on the plane F_4, infinitely near O_1'.

From this discussion of the different systems of curves belonging to $|C_0|$, one concludes that the surface F has at the point O_4' a double bi-planar point, which in turn is infinitely near two other bi-planar points, on F_2 and F_3, respectively, the latter being ordinary.

<u>Images</u> <u>of</u> <u>Curves</u> $|\underline{C}_1|$ <u>and</u> $|\underline{C}_6|$ <u>at</u> $\underline{O}_4'$.

The equation of the system of curves $|C_1|$ is

$$u_3x_2^6x_1 + u_2x_2^4x_1^2x_3 + u_1x_2^2x_1^3x_3^2 + u_4x_2x_3^6 + u_0x_1^4x_3^3 = 0 .$$

Applying the transformation W, five times, or

$$x_1 : x_2 : x_3 = z_1z_3^5 : z_2^6 : z_2^5z_3$$

once, gives

$$(32) \qquad z_2^{21}(u_3z_1 + u_4z_3) + u_2z_2^{14}z_1^2z_3^6 + u_1z_2^7z_1^3z_3^{12} + u_0z_1^4z_3^{18} = 0.$$

The curves (32), therefore, pass simply through the point $(z_1 = z_3 = 0)$, with variable tangent

$$u_3z_1 + u_4z_3 = 0 .$$

The curves $|C_1|$ pass simply through O_{23}, O_{233}, O_{2333}, O_{23333}, and O_{233333}, with a simple variable point in the first order neighborhood of O_{233333}.

Substituting $z_3 = kz_1$ in (32), gives

$$z_2^{21}(u_3z_1 + u_4kz_1) + u_2k^6z_2^{14}z_1^8 + u_1k^{12}z_2^7z_1^{15} + u_0k^{18}z_1^{22} = 0$$

or

$$z_2^{21}(u_3 + u_4k) + u_2k^6z_2^{14}z_1^7 + u_1k^{12}z_2^7z_1^{14} + u_0k^{18}z_1^{21} = 0 .$$

Letting z_1 approach zero, gives

$$u_3 + u_4k = 0 . \tag{33}$$

The value of k in (33), substituted into equation (24), will locate the point on the line a_{233333}, which point corresponds to the point of $|C_1|$ in the first order neighborhood of O_{233333}. The coordinates of this point are

$$(0,0,0,u_4,0,-u_3),$$

and the equations for the tangent line to the curves $|\Gamma_1|$ at O_4' are

$$u_3X_3 + u_4X_5 = 0 , \qquad X_0 = X_1 = X_2 = 0 . \tag{34}$$

The projection of the curves $|\Gamma_1|$ from O_4' on the surface F_2 passes through this point on the line a_{233333}.

Thus, the curves $|\Gamma_1|$ pass simply through the point O_4' and have (34) for their tangent. This tangent lies on the plane

$$X_0 = X_1 = X_2 = 0 ,$$

which is one of the two tangent planes to F at O_4'.

By the existing harmonic homology, one can immediately infer that the curves $|\Gamma_6|$ pass simply through

O_4'. Their tangent equations at this point are

(35) $\quad u_0X_0 + u_4X_3 = 0 , \qquad X_5 = X_1 = X_2 = 0 .$

This tangent lies on the second plane

$$X_5 = X_1 = X_2 ,$$

which is tangent to F at O_4'.

<u>Images of Curves $|C_2|$ and $|C_5|$ at O_4'.</u>

Consider the system $|C_2|$, where one has

$$u_2x_2^5x_1^2 + u_1x_2^3x_1^3x_3 + u_4x_2^2x_3^5 + u_0x_2x_1^4x_3^2 + u_3x_1x_3^6 = 0 ,$$

Apply the transformation W, two times. It follows that

$$u_2z_2^6z_1^2 + u_1z_2^3z_1^3z_3^2 + u_4z_2^5z_3^3 + u_0z_1^4z_3^4 + u_3z_2^2z_1z_3^5 = 0$$

and

$$u_2z_2^8z_1^2 + u_1z_2^4z_1^3z_3^3 + u_4z_2^9z_3 + u_0z_1^4z_3^6 + u_3z_2^5z_1z_3^4 = 0 .$$

Applying next the transformation R, one time, gives

$$u_2z_2^{14}z_1 + u_1z_2^7z_1^5z_3^3 + u_4z_2^{14}z_3 + u_0z_1^9z_3^6 + u_3z_2^7z_1^4z_3^4 = 0$$

or

(36) $\quad z_2^{14}(u_2z_1 + u_4z_3) + u_1z_2^7z_1^5z_3^3 + u_0z_1^9z_3^6 + u_3z_2^7z_1^4z_3^4 = 0.$

Thus, the curves (36) have a simple point at $(z_1 = z_3 = 0)$, the variable tangent being

$$u_2z_1 + u_4z_3 = 0 .$$

The curves $|C_2|$ pass simply through O_{2331} and O_{233}, and doubly through O_{23}. They have a simple variable point in the first order neighborhood of O_{2331}.

After substituting $z_3 = kz_1$, in equation (36), and letting z_1 approach zero, one gets

$$z_2^{14}(u_2z_1 + u_4kz_1) + u_1k^3z_2^7z_1^8 + u_0k^6z_1^{15} + u_3k^4z_2^7z_1^8 = 0$$

or

$$z_2^{14}(u_2 + u_4k) + u_1k^3z_2^7z_1^7 + u_0k^6z_1^{14} + u_3k^4z_2^7z_1^7 = 0$$

and

$$u_2 + u_4k = 0 \,. \tag{37}$$

The value of k in (37), substituted in equation (27), will locate the point on the line b_{2331}, which point corresponds to the point of $|C_2|$ in the first order neighborhood of O_{2331}. The coordinates of this point are

$$(0,\ 0,\ u_4,\ 0,\ 0,\ -u_2) \,.$$

It is concluded from this analysis that the projection of $\|\Gamma_2\|$, from the straight line $O_4'\,O_3'$ onto the surface F_3, passes through this point on the line b_{2331}. The projection of $\|\Gamma_2\|$, from O_4' onto F_2, passes simply through O_3', and has for tangent line

$$u_2X_2 + u_4X_5 = 0; \qquad X_0 = X_1 = X_4 = 0$$

at the point O_3'.

It follows that the curves $|\Gamma_2|$ pass simply through O_4'. Its tangent line at this point is

$$X_0 = X_1 = X_2 = X_5 = 0 \ .$$

When the curves $|\Gamma_2|$ are projected from O_4' onto the surface F_2, a twisted fifth order system of curves appear, having the equations

$$\left\| \begin{matrix} X_0X_5 & X_1 & X_2 & (-u_4X_5) \\ X_1^2 & X_2 & X_3 & (u_0X_1 + u_1X_2 + u_2X_3 + u_3X_5) \end{matrix} \right\| = 0,$$

$$X_4 = 0 \ .$$

By the harmonic homology, it is immediately established that to the point infinitely near O_{2113} on the curves $|C_5|$ corresponds on the line b_{2113} the point

$$(0,\ 0,\ u_1,\ 0,\ 0,\ -u_4) \ .$$

The projection of the curves $|\Gamma_5|$, from the straight line $O_4'\ O_3'$ on the surface F_3, passes through this point on the line b_{2113}.

The projection of $|\Gamma_5|$, from O_4' on F_2, passes simply through O_3', and has for tangent line

$$u_4X_2 + u_1X_0 \ , \qquad X_5 = X_1 = X_4 = 0$$

at the point O_3'.

The curves $|\Gamma_5|$ pass simply through O_4', and the

tangent line at this point is

$$x_0 = x_1 = x_2 = x_5 = 0 \, .$$

When the curves $|\Gamma_5|$ are projected from O_4' onto the surface F_2, they become a twisted fifth order system of curves, having for their equations

$$\left\| \begin{matrix} x_0x_5 & x_1 & x_2 & (-u_1x_0) \\ x_1^2 & x_2 & x_3 & (u_0x_0 + u_2x_1 + u_3x_2 + u_4x_3) \end{matrix} \right\| = 0$$

$$x_4 = 0 \, .$$

The systems of curves $|\Gamma_2|$ and $|\Gamma_5|$ have, therefore, the common tangent line

$$x_0 = x_1 = x_2 = x_5 = 0$$

at the point O_4'. This tangent line is the line of intersection of the two tangent planes to the surface F at the branch point O_4'.

<u>Images of Curves $|C_3|$ and $|C_4|$ at O_4'.</u>

Finally, one takes up the curves $|C_3|$. Their equation is

$$u_2x_2^4x_1^3 + u_4x_2^3x_3^4 + u_1x_2^2x_1^4x_3 + u_3x_2x_1x_3^5 + u_0x_1^5x_3^2 = 0 \, .$$

Applying the transformations, W once and R twice, give

$$u_2z_2^6z_1^3 + u_4z_2^8z_3 + u_1z_2^3z_1^4z_3^2 + u_3z_2^5z_1z_3^3 + u_0z_1^5z_3^4 = 0$$

and

$$u_2z_2^{10}z_1^2 + u_4z_2^{11}z_3 + u_1z_2^5z_1^5z_3^2 + u_3z_2^6z_1^3z_3^3 + u_0z_1^8z_3^4 = 0$$

and

$$u_2z_2^{14}z_1 + u_4z_2^{14}z_3 + u_1z_2^7z_1^6z_3^2 + u_3z_2^7z_1^5z_3^3 + u_0z_1^{11}z_3^4 = 0$$

or

$$(38)\quad z_2^{14}(u_2z_1 + u_4z_3) + u_1z_2^7z_1^6z_3^2 + u_3z_2^7z_1^5z_3^3 + u_0z_1^{11}z_3^4 = 0.$$

Thus, the curves (38) have a simple point at $(z_1 = z_3 = 0)$, the variable tangent being

$$u_2z_1 + u_4z_3 = 0.$$

The curves $|C_3|$ pass simply through O_{23}, O_{231}, O_{2311}, and have a simple variable point in the first order neighborhood of O_{2311}.

Replace z_3 with kz_1 in equation (38), and then let z_1 approach zero. It follows that

$$z_2^{14}(u_2 + u_4k) + u_1k^2z_2^7z_1^7 + u_3k^3z_2^7z_1^7 + u_0k^4z_1^{14} = 0$$

and

$$(39)\qquad u_2 + u_4k = 0.$$

The value of k in (39), substituted into equation (30), will locate the point on the line c_{2311}, which point

corresponds to the point of $|C_3|$ in the first order neighborhood of O_{2311}. The coordinates of this point are

$$(0,\ u_4,\ 0,\ 0,\ 0,\ -u_2)\ .$$

One concludes from this investigation, that the projection of $\left|\Gamma_3\right|$, from the plane $O_4'\ O_3'\ O_2'$ onto the plane F_4, passes through this point on the line c_{2311}. The projection of $\left|\Gamma_3\right|$, from the line $O_4'\ O_3'$ on F_3, passes simply through O_2', along the tangent line

$$u_2X_1 + u_4X_5 = 0, \qquad X_0 = X_3 = X_4 = 0\ .$$

The equation of these curves on F_3 are

$$\left\|\begin{matrix} X_1 & X_0X_5 & (-u_1X_0X_1 - u_2X_0X_2) \\ X_2 & X_1^2 & (u_3X_1^2 + u_4X_1X_2 + u_0X_0X_1) \end{matrix}\right\| = 0$$

$$X_3 = X_4 = 0\ .$$

Thus, the curves $\left|\Gamma_3\right|$ have a simple point at O_4'. The tangent line is the intersection of the planes

$$X_0 = X_1 = X_5 = 0$$

and

$$X_0 = X_1 = X_2 = 0,$$

which is the line

$$X_0 = X_1 = X_2 = X_5 = 0\ .$$

By the harmonic homology, one concludes also that to the point infinitely near O_{2133} on the curves $|C_4|$, corresponds on the line c_{2133} the point

$$(0,\ u_1,\ 0,\ 0,\ 0,\ -u_4)\ .$$

The projection of $|\Gamma_4|$, from the plane $O_4'\ O_3'\ O_2'$ on the plane F_4, passes through this point on the line c_{2133}. The projection of $|\Gamma_4|$, from the line $O_4'\ O_3'$ onto F_3, passes simply through O_2' along the tangent line

$$u_4x_1 + u_1x_0 = 0, \qquad x_3 = x_4 = x_5 = 0\ .$$

The equations of these curves on F_3 are

$$\left\|\begin{matrix} x_1 & x_0x_5 & (-u_3x_1x_5 - u_4x_2x_5) \\ x_2 & x_1^2 & (u_0x_1^2 + u_1x_1x_2 + u_2x_1x_5) \end{matrix}\right\| = 0$$

$$x_3 = x_4 = 0,$$

and the curves $|\Gamma_4|$ have a simple point at O_4'. The tangent line is the intersection of the planes

$$x_0 = x_1 = x_5 = 0$$

and

$$x_5 = x_1 = x_2 = 0,$$

which is the line

$$x_0 = x_1 = x_2 = x_5 = 0\ .$$

It follows, therefore, that the systems of curves

$|\Gamma_3|$ and $|\Gamma_4|$ pass simply through the branch point O_4'. The tangent line is the line of intersection of the two tangent planes to F at the point O_4'.

<u>Summary.</u>

The results of this chapter show that the image of a plane cyclic involution of period seven may be taken as a surface of order seven in a linear space of five dimensions. The surface has two four-tuple branch points, whose tangent cones are formed by a third order cone and a plane. The surface has also a third branch point, which is double bi-planar and infinitely near two double bi-planar points not on the surface.

There exist on the surface six linear systems (∞^4) of twisted septic curves. One system, $|\Gamma_1|$, passes triply through one of the four-tuple points, with the three tangent lines lying on the cubic cone; $|\Gamma_1|$ also passes simply through the other four-tuple point, with the tangent lying in the tangent plane at that point. Finally, $|\Gamma_1|$ passes simply through the double bi-planar point, with its tangent line in one of the two tangent planes at this point.

A second system, $|\Gamma_6|$, has the same characteristics as $|\Gamma_1|$, only the roles of the two four-tuple points and the two tangent planes at the double bi-planar point are reversed.

A third system, $|\Gamma_2|$, passes triply through one

of the two four-tuple points, with two tangent lines on the cubic cone and one on the tangent plane. It passes simply through the other four-tuple point, with its tangent line on the cubic cone at that point. Finally, $|\Gamma_2|$ passes simply through the double bi-planar point, with its tangent line the line of intersection of the two tangent planes at that point.

A fourth system, $|\Gamma_5|$, has results analogous to $|\Gamma_2|$, except that the roles of the two four-tuple points are reversed.

A fifth system, $|\Gamma_3|$, passes doubly through one of the two four-tuple points, with the two tangent lines lying on the cubic cone. It passes doubly through the other four-tuple point, with one tangent line on the cubic cone and the other on the tangent plane. Finally, $|\Gamma_3|$ passes simply through the double bi-planar point. It has for its tangent line the line of intersection of the two tangent planes at this point.

A sixth and final system $|\Gamma_4|$ has the same properties as $|\Gamma_3|$, with the roles of the two four-tuple points reversed.

When the systems of curves, $|\Gamma_1|$ and $|\Gamma_6|$, are projected from one of the two four-tuple points onto a hyperplane, two new systems result, which are twisted fourth order curves.

When the systems $|\Gamma_2|$ and $|\Gamma_5|$ are projected from the double bi-planar point, and when the systems $|\Gamma_3|$ and $|\Gamma_4|$ are projected from the line connecting this point with its adjacent double bi-planar point, O_3', both onto respective hyperplanes, twisted fifth order curves result.

CHAPTER VIII

CONCLUSION

When one compares the findings of Chapters III, V, and VII, it is interesting to note that the three branch points on the image surfaces can be classified into two types. Two of the branch points fall into one of these types, while the third branch point belongs to the other type. The points of the first type are double, triple, or four-tuple, when the involutions are of periods three, five, or seven, respectively. The tangent cones are of the degenerate type. They are formed by a first order cone (plane) and a plane, a second order cone and a plane, and a third order cone and a plane, respectively.

The third branch point, being of the second type, is always a double bi-planar point.

It is also noted that the order of the image surfaces is the same as the period of the involution considered. Likewise, the images of the systems of invariant plane curves which pass through the base points of the reference triangle, are twisted curves on the image surface and they are of the same order as the period of the involution considered.

The invariant space curves were supported by surfaces in S_3, while invariant planes curves were supported by a plane. Furthermore, these surfaces were all distinct. Hence, it is apparent that one would not expect the findings in Chapters II, IV, and VI to be comparable to those for the plane curves.

However, one common topic included the study of the behavior of those invariant curves on the tangent planes to the surfaces at the simple points of coincidences.

The determination of the location of the perfect points, adjacent to the invariant points on these surfaces in S_3, agrees with a certain pattern which is discussed in more detail in J. C. Morelock's dissertation [10].

While this dissertation was being prepared, considerable discussion came up concerning the best methods for determining the order of surfaces and space curves. The method used by M^{lle}Dessart [1] involved the determination of the number of planes, which were common to the set of independent equations defining the surface, but which did not satisfy the set of dependent equations also necessary to accurately describe the surface. The number of such planes was subtracted from the possible maximum order (the product of the degrees of each independent equation) in order to arrive at the actual order of the surface.

The author had the pleasure of meeting the well-known algebraic geometrist, A. B. Coble, at a sectional meeting of The Mathematical Association of America (1952). When asked for his method of determining the order of the surface F in Chapter VII, Dr. Coble submitted the following:

$$F, \quad \left\| \begin{matrix} x_1 & x_2 & x_3 & x_0x_5 \\ x_2 & x_3 & x_4 & x_1^2 \end{matrix} \right\| = 0$$

The matrix of the first three columns defines in S_5 a mani- 3-parameter hypersurface fold M_3, whose parametric equations are

$$x_1 = t^3, \qquad x_4 = 1,$$
$$x_2 = t^2, \qquad x_0 = x_0,$$
$$x_3 = t, \qquad x_5 = x_5,$$

where t, x_0, and x_5 are three non-homogeneous parameters.

This M_3 is cut by the remaining determinant of the matrix in M_2, determined on M_3 by the condition

$$C, \qquad x_0x_5 - t^7 = 0$$

The order of M_2 is the number of points in which it is met by a generic S_3 in S_5. A generic S_3 in S_5 has equations

$$x_0 = a_0x_1 + 3a_1x_2 + 3a_2x_3 + a_3x_4$$

and

$$X_5 = b_0X_1 + 3b_1X_2 + 3b_2X_3 + b_3X_4$$

Applying these to M_3 as conditioned by C, one gets

$$D = (a_0t^3+3a_1t^2+3a_2t+a_3)(b_0t^3+3b_1t^2+3b_2t+b_3) - t^7 = 0,$$

and thus a septic in t. For each t, which is a root of D, there is a point of the generic S_3 on M_2, whence M_2 is of order seven.

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BIOGRAPHICAL SKETCH

Svend Theodore Gormsen was born in the small village of Harndrup, Denmark, on February 24, 1909. He attended elementary and secondary schools there, graduating from high school in 1925.

Two years later, in April 1927, he came to the United States, where he took up permanent residence.

After working at odd jobs for two years, and studying radio at night at the Chicago Radio Institute, he was employed as a shipboard radio operator for seven seasons.

In the fall of 1931 he entered Ohio State University, where he graduated in 1935 with a degree of B. S. in Education, majoring in mathematics and minoring in physics and chemistry.

The next eight years found the author teaching mathematics and science in high schools at Blasdell, N. Y., Liberty, N. Y., and Lakewood, Ohio.

From 1943-1946 he served in the U. S. Navy as a Radar Design Project Engineer. He was placed on inactive duty in May 1946 with the rank of Lieutenant Commander.

After his active military duty he was an instructor of mathematics at Syracuse University until February 1947, at which time he assumed a similar position at the

University of Florida.

The author received his M. S. degree at this institution in 1949 and now holds the rank of Assistant Professor in its Department of Mathematics.

He is an active member of the Mathematical Association of America, Pi Mu Epsilon, and Phi Kappa Phi.

He married Miss Virginia M. Warden of Zanesville, Ohio, in 1936. She is also a graduate of Ohio State University. They have two children, Jimmy (born in 1937) and Gayle (born in 1946).

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

January 31, 1953

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